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MATHEMATICS

Some Theorems on the Sweeping in Banach Spaces

by

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Presented by K. BORSUK on June 18, 1959

For arbitrary metric spaces X and Y denote by Y^X the set of all mappings of X into Y and by $X \times Y$ the Cartesian product of X and Y. If $X_0 \subset X$ and $f \in Y^X$, then $f|X_0$ will denote the partial mapping of f defined in X_0, i.e. the mapping f_0 defined in X_0 by the formula $f_0(x) = f(x)$; we shall say that f is an extension of f_0 over X and we then write $f_0 \subset f$.

Let X_0, X_1 be closed subsets of the space X. A continuous mapping $f \in X^{X_0 \times I}$ (I — denotes the closed interval [01]) is said to be a *continuous deformation* of X_0 into X_1 in the space X if

$$f(x, 0) = x \quad \text{for every} \quad x \in X_0$$
$$f(x, 1) \in X_1 \quad \text{for every} \quad x \in X_0 .$$

A point $y_0 \in X \setminus X_0$ is said to be *swept* by the continuous deformation $f \in X^{X_0 \times I}$ of X_0 into X_1 in the space X if the equation

$$y_0 = f(x, t)$$

has at least one solution $x_0 \in X_0$, $t_0 \in I$.

A set $U \subset X \setminus X_0$ is said to be swept by a continuous deformation $f \in X^{X_0 \times I}$ of X_0 into X_1 in the space X, if every point $y_0 \in U$ is swept by f.

In 1931 K. Borsuk proved the following theorem on the sweeping [2]:

Let X be a compact subset of the n-dimensional Euclidean space E_n and let $f \in E_n^{X \times I}$ be a continuous deformation of X into a single-point set $\{x^*\}$ in the space E_n. Then every bounded component U of $E_n \setminus X$ is swept by the deformation f.

In this paper we shall give an extension of the above Borsuk theorem of the case of an arbitrary Banach space. The proof of the main theorem is quite elementary and does not use homology notions. As an application we obtain the well-known Birkhoff-Kellog theorem [1], concerning the existence of an invariant direction for a completely continuous operator in Banach space.

Preliminaries

1. Denote by E_∞ an arbitrary Banach space and by P_∞ the space E_∞ without the point 0; if a point $x_0 \in E_\infty$ and r are positive numbers, then we denote by $V_\infty(x_0, r)$ the closed full sphere in E_∞ with centre x and radius r and its boundary by $S_\infty(x, r)$. If $X \subset E_\infty$, then we denote by conv (X) the closed convex hall of X in E_∞; if λ is a real number, then we denote by λX the set of all points $\lambda x, x \in X$.

A mapping $F: X \to E_\infty$ of a metric space X into E_∞ is called *completely continuous* on X if the image $F(X)$ is compact in E_∞ *).

The following proposition is known **):

(1, 1). *If X_0 is a closed subset of a metric space X and $F_0: X_0 \to E_\infty$ is a completely continuous mapping on X_0, then there exists a completely continuous mapping $F: X \to E_\infty$ such that:*

1° $F(X) \subset \mathrm{conv}(F_0(X_0))$,

2° $F_0 \subset F$.

2. Let now X be a closed and bounded subset of E_∞.

The mapping $f: X \to E_\infty$ will be called *completely continuous field on X* if it can be represented in the form:

$$f(x) = x - F(x),$$

where the mapping $F: X \to E_\infty$ is completely continuous on X.

The set of all completely continuous fields defined on X with the values in some $Y \subset E_\infty$ will be denoted by $\mathfrak{C}(Y^X)$.

Definition 1. We shall say that two completely continuous fields $f, g \in \mathfrak{C}(P_\infty^X)$ are homotopic in $\mathfrak{C}(P_\infty^X)$ (we shall write $f \simeq g$ in $\mathfrak{C}(P_\infty^X)$) if there exists a mapping $H: X \times I \to E_\infty$ such that:

1° H is completely continuous on $X \times I$,

2° the mapping $h(x, t) = x - H(x, t) \neq 0$ for $x \in X$ and $t \in I$,

3° $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ for every $x \in X$.

We shall use the following Homotopy Extension Theorem [4]:

(2, 1). *Let X_0 be a closed subset of $X \subset E_\infty$, $f_0, g_0 \in \mathfrak{C}(P_\infty^{X_0})$ and $f_0 \simeq g_0$ in $\mathfrak{C}(P^{X_0})$. If $f_0 \subset f \in \mathfrak{C}(P_\infty^X)$, then there exists $g \in \mathfrak{C}(P_\infty^X)$ such that $g_0 \subset g$ and $g \simeq f$ in $\mathfrak{C}(P_\infty^X)$.*

The following proposition is an immediate consequence of the Schauder Fixed Point Theorem [7]:

(2, 2). *Let X be a closed bounded domain in E_∞ and let X_0 be the boundary of X. If x_0 is an interior point of X, then there exists no completely continuous field $f \in \mathfrak{C}(P_\infty^X)$ such that $f(x) = x - x_0$ for every $x \in X_0$.*

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*) We shall denote completely continuous mappings in the sequel by the capital letters F, F_0, G, H etc.

**) Proposition (1, 1) is a special case of the Extension Theorem proved by J. Dugundji [3], but it can easily be deduced from Tietze's Extension Theorem.

The main theorem

3. DEFINITION 2. Let X_0 and X_1 be closed bounded subsets of E. A continuous deformation $f \in E_\infty^{X_0 \times I}$ of X_0 into X_1 in the space E_∞ is said to be *completely continuous* if it can be represented in the form:

$$f(x, t) = x - F(x, t),$$

where the mapping $F: X_0 \times I \to E_\infty$ is completely continuous on $X_0 \times I$.

The proof of the main theorem is based on the following

LEMMA. *Let X be a closed bounded subset of E_∞, d — the diameter of X, y_0 — a point belonging to a bounded component U of $E_\infty \setminus X$. Let $f \in \mathfrak{C}\big((V_\infty(z_0, p))^X\big)$, where the point z_0 and the number p are such that $|y_0 - z_0| > p + 2d$.*

In this case the completely continuous field

$$g(x) = f(x) - y_0 = x - F_0(x) - y_0, \quad x \in X$$

can be extended to some field $g^* \in \mathfrak{C}(P^{X \cup U})$ over $X \cup U$.

Proof. From the inequalities

$$\|F(x) - (y_0 - z_0)\| = \|x - f(x) - y_0 - z_0\| \leqslant \|f(x) - z_0\| + \|x - y_0\| < p + d$$

it follows that, for every $x \in X$,

$$F(x) \in V_\infty(y_0 - z_0, p + d).$$

By (1,1) the completely continuous mapping $F_0(x)$ can be extended to a completely continuous mapping $F^*(x)$ over $X \cup U$ with the values in $V_\infty(y_0 - z_0, p + d)$.

Since, for every $x \in X \cup U$,

$$F^*(x) \in V_\infty(2y - z_0, p + d)$$

and

$$U \cap V_\infty(2y_0 - z_0, p + d) \subset V_\infty(y_0, d) \cap V_\infty(2y_0 - z_0, p + d) = \emptyset \text{ *)}$$

it is clear that if we define $g^*(x)$ by

$$g^*(x) = x - F^*(x) - y_0 \quad \text{for} \quad x \in X \cup U$$

then we obtain a completely continuous field $g^* \in \mathfrak{C}(P_\infty^{X \cup U})$ which will be the desired extension of g over $X \cup U$.

4. The main result of this paper is as follows:

THEOREM 1. *Let X be a bounded closed subset of the Banach space E_∞ and $f \in E_\infty^{X \times I}$, $f(x, t) = x - F(x, t)$ a completely continuous deformation of X into a closed subset $Y \subset E_\infty$ in the space E_∞. If a point y_0 belongs to a bounded component of $E_\infty \setminus X$ and to the unbounded component of $E_\infty \setminus Y$, then y_0 is swept by f.*

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*) $\emptyset$ — denotes the empty set.

Proof. Suppose, on the contrary, that y_0 is not swept by f, i. e. the equation $y_0 = f(x, t)$ has no solution. In this case, if we put for $x \in X$

$$f_0(x) = f(x, 0) - y_0 = x - y_0,$$
$$f_1(x) = f(x, 1) - y_0 = x - F(x, 1) - y_0,$$

we obtain completely continuous fields $f_0, f_1 \in \mathfrak{C}(P_\infty^X)$, homotopic in $\mathfrak{C}(P_\infty^X)$

$$(1) \qquad f_0 \simeq f_1 \quad \text{in} \quad \mathfrak{C}(P_\infty^X).$$

Let z_1 be a point belonging to the unbounded component of $E_\infty \setminus Y$ such that $\|z_1\| > 2d + p$, where the number d is the diameter of X and $p = \sup_{x \in Y} \|x\|$.

Since y_0 belongs to the same component, there exists a continuous function $\varphi(t)$ of the real parameter t, $0 \leqslant t \leqslant 1$ with the values in $E_\infty \setminus Y$ such that

$$\varphi(0) = y_0, \quad \varphi(1) = z_1.$$

Let us put

$$g(x, t) = x - F(x, 1) - \varphi(t) \quad \text{for} \quad x \in X \quad \text{and} \quad t \in I,$$
$$g_0(x) = g(x, 0), \quad g_1(x) = g(x, 1) \quad \text{for} \quad x \in X,$$
$$h(x) = g_1(x) + y_0 \quad \text{for} \quad x \in X,$$
$$z_0 = y_0 - z_1.$$

We have $g_0, g_1 \in \mathfrak{C}(P_\infty^X)$ and

$$(2) \qquad g_0 \simeq g_1 \quad \text{in} \quad \mathfrak{C}(P_\infty^X).$$

Since $\|h(x) - z_0\| = \|g_1(x) + y_0 - z_0\| = \|g_1(x) + z_1\| = \|x - F(x, 1)\| \leqslant p$, we obtain that, for every $x \in X$, $h(x) \in V_\infty(z_0, p)$, i.e. $h \in \mathfrak{C}\big(\big(V_\infty(z_0, p)\big)^X\big)$.

Since $\|z_0 - y_0\| = \|z_1\| > 2d + p$, then applying the Lemma to h we infer that the completely continuous field

$$g_1(x) = h(x) - y_0$$

can be extended over $X \cup U$ to a field $g \in \mathfrak{C}(P_\infty^{X \cup U})$.

Since $f_1(x) = g_0(x)$ for every $x \in X$, it follows by (2,1) from (1) and (2) that the completely continuous field f_0 can be extended over $X \cup U$ to some completely continuous field $f^* \in \mathfrak{C}(P_\infty^{X \cup U})$. The latter contradicts (2,2) which completes the proof.

As an immediate consequence of Theorem 1 we obtain.

Theorem 2. *Let X be a bounded closed subset of E_∞ and let $f \in E_\infty^{X \times I}$ be a completely continuous deformation of X into in the space E_∞. If a bounded component U of $E_\infty \setminus X$ is contained in an unbounded component of $E_\infty \setminus Y$, then U is swept by the deformation f.*

Application to the characteristic value problem

5. Let $X_0 \subset E$ and $F_0: X_0 \to E_\infty$ be a completely continuous mapping on X_0. We shall say that the real number μ_0 is a *characteristic value* for a mapping F_0 if the equation

$$x = \mu_0 F_0(x) \tag{3}$$

has at least one solution $x_0 \in X_0$.

As an application of the Theorem 1 we shall prove the following:

Corrolary. (Birkhoff-Kellog Theorem) *Let X_0 be the boundary of a bounded domain $X \subset E$, which contains the point 0. If a completely continuous mapping $F_0: X_0 \to E_\infty$ is such that*

$$\|F_0(x)\| > a > 0, \quad \textit{for each} \quad x \in X_0, \tag{4}$$

then F_0 has at least one characteristic value μ_0, i.e. *the Eq.* (3) *has at least one solution for some μ_0* *).

Proof. Let $S_\infty = S_\infty(0, 1)$ be the unit sphere in E_∞ and let us put

$$F(x) = -\frac{F_0(x)}{\|F_0(x)\|} \quad \text{for} \quad \text{each } x \in X_0. \tag{5}$$

It follows from (4) that the mapping $F: X_0 \to S_\infty$ is completely continuous on X_0. The closure $\overline{F(X_0)}$ is a compact subset of S_∞. Since S_∞ is not compact, it follows that there is a point $x^* \in S_\infty$ and a positive number ε such that the set $V_\infty = V_\infty(x^*, \varepsilon)$ does not intersect $\overline{F(X_0)}$.

Let us consider the mapping f_λ defined on X_0 by the formula

$$f_\lambda(x) = x + \lambda F(x), \quad \lambda > 0.$$

Since the values of $\lambda F(x)$ lie in λS_∞ we have

$$(\lambda F)(X_0) \cap \lambda V_\infty = \theta; \tag{6}$$

since the set X_0 is bounded we have

$$\|\lambda F(x) - f_\lambda(x)\| \leqslant \|x\| < K \quad \text{for} \quad x \in X_0. \tag{7}$$

From (6) and (7) it follows that if a number λ is sufficiently large $\lambda \geqslant \lambda_0$, then the closed set $Y = f_{\lambda_0}(X_0)$ does not intersect the half-ray μx^*, $0 \leqslant \mu < +\infty$ i.e. Y does not separate E_∞ between 0 and infinity.

Let us put

$$f(x, t) = x + t\lambda_0 F(x) \quad \text{for} \quad x \in X_0 \text{ and } 0 \leqslant t \leqslant 1.$$

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*) For Birkhoff-Kellog Theorem see [1], [6] p. 191 and also [5] p. 187. In [1] only the case of the space $C(0,1)$ of continuous functions in the interval [01] was considered. The general case of arbitrary Banach spaces was considered in [6] and [5]. The proofs in [6] and [5] make an essential use of the Poincaré Theorem on the vector fields on the sphere of an even dimension.

Obviously, $f \in E_\infty^{X_0 \times I}$ is a completely continuous deformation of X_0 into Y in the space E_∞ and hence, by Theorem 1, the point 0 is swept by f, i.e.

$$x_0 + \lambda_0 F(x_0) = 0 \quad \text{for some} \quad x_0 \in X_0 \text{ and } t_0 \in I\,. \tag{8}$$

Putting $\mu_0 = \dfrac{t_0 \lambda_0}{\|F_0(x_0)\|}$ we obtain from (5) and (8) $x_0 = \mu_0 F_0(x_0)$ and this completes the proof.

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MATHEMATICS

On the Categoricity in Power $\leqslant \aleph_0$

by

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Presented by A. MOSTOWSKI on June 29, 1959

In the present paper some results are given concerning the notion of categoricity in power, introduced by J. Łoś [1] and R. L. Vaught [4]. The aim of this note is the characterization of the categorical theories in power $\leqslant \aleph_0$.

1. Let T be an arbitrary elementary theory, for which the system of primitive notions is a certain fixed system (finite or denumerable) of predicates. It is assumed, that the logic of the theory is the lower functional calculus with identity. Non-contradiction and completeness of the theory T are, further, assumed. By $x_1, x_2, \ldots$ will be denoted the individual variables; small Greek letters $\alpha, \beta, \ldots, \varphi, \psi, \ldots$ will stand for the formulae of the theory T; we shall also write $\alpha(x_1, \ldots, x_n)$ in order to point to the fact that only the variables $x_1, \ldots, x_n$ are free in the formula α.

The Boolean algebra of the formulae of the theory T is formed in the known manner: we denote namely by $[\alpha]$ a class of formulae equivalent to the formula α($[\alpha] \overset{\mathrm{df}}{=} \{\beta: {}''\alpha \equiv \beta''$ is the theorem $T\}$), whereas the inclusion $[\alpha] \subset [\beta]$ means that $''\alpha \to \beta''$ is the theorem T. Then, obviously, logical functors are interpreted as Boolean operations of the algebra F. For example: $[\alpha \vee \beta] = [\alpha] \vee [\beta]$; $\vee$ on the left means an alternative, on the right — summation, and so on.

By F_n we will denote a set of classes $[\alpha]$ such that α depends only on $x_1, \ldots, x_n$ (i.e. we may write $\alpha = \alpha(x_1, \ldots, x_n)$). Obviously, F_n is an subalgebra of the algebra F.

The theory T, being non-contradictory, has its models; we assume throughout that the predicate of identity is interpreted in the model in a natural manner by means of the relation of the identity. Therefore, the sign $'' = ''$ will denote the predicate "equality" as well as the "true" equality and this should not lead to any noxious ambiguities.

If A is a model of the theory T, $a_1, \ldots, a_n$ being its elements and $\alpha(x_1, \ldots, x_n)$ any arbitrary formula, then

$$\alpha_A(a_1, \ldots, a_n) = 1 \quad (\text{or } 0)$$

will mean that the elements $a_1, \ldots, a_n$ in the model A substituted for free variables $x_1, \ldots, x_n$ satisfy (or not) the formula α.

The categoricity of the theory T in power $\leqslant \aleph_0$ means — as is known — that each two models of the power $\leqslant \aleph_0$ are isomorphic.

2. The main result of the present paper may be stated as follows:

THEOREM. *The theory T is categorical in power $\leqslant \aleph_0$ if and only if for every $n = 1, 2, \ldots$ the algebra F_n is finite.*

Proof. There are two cases possible: either all propositions of the form $\sum\limits_{x_1, \ldots, x_s} \bigcap\limits_{i<j\leqslant s} (x_i \neq x_j)$ are theorems of the theory T, or they are not; it is easily seen, that in the first case all models are finite, possess an equal number of elements and are isomorphic; then the power of the set F_n is not greater than n^m (where m stands for the number of elements of the model). So, we may assume in the sequel that the models of the theory T are infinite.

SUFFICIENCY. Let two denumerable models A and B be given. We order both of them (in an arbitrary but fixed manner) so as to have an ordering of the ω-type.

By induction we define now two infinite sequences $\{a_n\}, \{b_n\}$ of elements from A and B, and a certain sequence of formulae possessing the following properties:

(i) $\alpha^{(n)} \big(= \alpha^{(n)}(x_1, \ldots, x_n)\big)$ is an atom of the algebra F_n,

(ii) $\alpha_A^{(n)}(a_1, \ldots, a_n) = 1$ for $n = 1, 2, \ldots$

(iii) $\alpha_B^{(n)}(b_1, \ldots, b_n) = 1$ for $n = 1, 2, \ldots$

Let us suppose that the construction was performed up to a certain point $n_0 (0 \leqslant n_0)$; if n_0 is even, we put

Def. 1. $a_{n_0+1} =$ the first element of the model A differing from $a_1, \ldots, a_{n_0}$;

$\alpha^{(n_0+1)} =$ is an atomic formula from the algebra F_{n_0+1} such that the relations (i) and (ii) hold for it (such a formula exists because of the finiteness of F_{n_0+1});

$b_{n_0+1} =$ the first element of the model B for which the relation (iii) holds.

Evidently, the proof of existence of the element b_{n_0+1}, possessing the properties mentioned above, is reduced to the proof that the formula

$$\beta(x_1, \ldots, x_{n_0}) \overset{\text{df}}{=} \sum_{x_{n_0+1}} \alpha^{(n_0+1)}(x_1, \ldots, x_{n_0})$$

is verified by the system $b_1, \ldots, b_{n_0}$. In order to prove it, let us remark that the alternative

$$[\alpha^{(n_0)}] \subset [\beta] \quad \text{or} \quad [\alpha^{(n_0)}] \subset [\widetilde{\beta}]$$

is valid.

The second eventuality is inadmissible, because — by inductive assumption — we have $\alpha_A^{(n_0)}(a_1, \ldots, a_n) = 1$, and in virtue of the definition of the formulae $\alpha^{(n_0+1)}$ and β we have also $\beta_A(a_1, \ldots, a_n) = 1$. Therefore, we have $[\alpha^{(n_0)}] \subset [\beta]$. Having in mind that, by inductive assumption, $a_B(b_1, \ldots, b_{n_0}) = 1$ we obtain finally the relation $\beta_B^{(n_0)}(b_1, \ldots, b_{n_0}) = 1$, which completes the proof.

If, however, n_0 is odd, we change the roles of the models A and B in the definition 1:

Def. 1′. $b_{n_0}+1 =$ the first element of the model B differing from $b_1, \ldots, b_{n_0}$;

$\alpha^{(n_0+1)}$ is an atomic formula from the algebra F_{n_0+1} such that the relations (i) and (iii) hold for it;

$a_{n_0+1} =$ the first element of the model A for which the relation (ii) holds.

Now, it is easy to complete the proof of existence of the isomorphism of models A and B.

There exist the following properties:

(1) for any arbitrary formula $\varphi(x_1, \ldots, x_n)$ the equality

$$\varphi_A(a_1, \ldots, a_n) = \varphi_B(b_1, \ldots, b_n)$$

holds.

This property follows immediately from the relations (i) — (iii) and from the obvious alternative

$$[\alpha^{(n)}] \subset [\varphi] \quad \text{or} \quad [\alpha^{(n)}] \subset [\widetilde{\varphi}]\,.$$

(2) The sequences $\{a_n\}$ and $\{b_n\}$ exhaust the sets A and B, respectively; this follows from the way of defining a_{n_0+1} and b_{n_0+1} and from the fact that the auxiliary order in A and B is of ω-type.

(3) The terms of the sequences $\{a_n\}$ and $\{b_n\}$ are different.

In fact, after the Definitions 1 and 1′, for $n < n_0+1$ and n_0 even, we have the inequalities $a_n \neq a_{n_0+1}$; for $n < n_0+1$ and n_0 odd, we have $b_n \neq b_{n_0+1}$. From these inequalities and from the property already proved (1), applied to the formulae φ of the form "$x_i \neq x_j$" all others inequalities may be obtained.

Thus, summing up (1)-(3), we may state, that the correspondence $a_n \leftrightarrow b_n$ is really an isomorphism of the models A and B.

Necessity. Let us now assume that for a certain n the algebra is infinite (hence, denumerable). There exists, therefore, a certain sequence of the formulae $\psi^k(x_1, \ldots, x_n)$ such that

$$[\psi^{(k)}] \subset [\psi^{(k+1)}] \quad \text{and} \quad [\psi^{(k)}] \neq [\psi^{(k+1)}] \quad \text{for} \quad k = 1, 2, \ldots$$

and

$$\text{(S)} \qquad \bigcup_{k=1}^{\infty} [\psi^{(k)}] = 1 ,$$

where 1 denotes the Boolean unit and $\bigcup$ – the infinite Boolean summation.

Then a denumerable model A (of the theory T) exists possessing the following property

(W) for arbitrary $a_1, \ldots, a_n \in A$ the equality $\psi_A^{(k)}(a_1, \ldots, a_n) = 1$, holds for any natural k.

The existence of such a model follows from an easy modification of the proof of Gödel's theorem given by Rasiowa and Sikorski [3]. We namely seek a maximal ideal of the algebra F, still preserving denumerable sums of this algebra corresponding to the existential quantifier and, moreover, preserving also the sum (S).

There exists also a denumerable model B not possessing the property (W). We enlarge namely the theory T by addition of an infinite sequence of propositions:

$$\widetilde{\psi}^{(k)}(\varkappa_1, \ldots, \varkappa_n) , \qquad k = 1, 2, \ldots$$

where $\varkappa_1, \ldots, \varkappa_n$ are new symbols, denoting individuals. The theory enlarged in this way is non contradictory. Let B be its model (Gödel's theorem again). This model, of course, does not possess the property (W). Thus, it is not isomorphic with the model A.

Remark. The proof of sufficiency was carried out assuming that the theory does not possess finite models. This assumption may be neglected as well. But, if so, in the inductive proceeding (Def. 1 and 1′) the sequences $\{a_n\}$ and $\{b_n\}$ could be finite. It should be additionally verified (it is not difficult) that even then the relation (2) holds.

The above results were communicated at the meeting of the Polish Mathematical Society, Toruń Branch, in 1955 (see also [2], p. 24).

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MATHEMATICS

On the Local Tests for Convergence of Fourier Series

by

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Several tests of various sorts for point-convergence of Fourier series are known [1]-[4]. Since the convergence at a point of Fourier series of an integrable function depends only on the specific properties of the function in the neighbourhood of the point, the local tests for convergence stand as certain conditions imposed on the function in the neighbourhood implying the convergence at the point. Not only when studying the Fourier series or, more generally, orthogonal series, but also in many other problems of analysis we have to deal with classes of functions (e.g. continuous functions) which are additionally supposed to satisfy some other local restrictions as e.g. the existence of a derivative or Lipschitz condition. Let us confine ourselves to a class $\mathcal{K}$ consisting of continuous functions.

The purpose of this paper is to show that, if the local condition $\boldsymbol{W}$, which is additionally imposed on the functions from $\mathcal{K}$, satisfies the system of postulates given below, then there exist in $\mathcal{K}$ functions which fail to satisfy $\boldsymbol{W}$ in any neighbourhood of every point belonging to a certain denumerable set (Theorem 3). This furnishes, if a suitable $\mathcal{K}$ is chosen, a proof that the convergency-tests for Fourier series of one certain type never give the necessary conditions for convergence.

1. In the sequel $\mathcal{K}$ will always denote a certain class of continuous functions in $\langle a, b\rangle$, and ξ — some value from $\langle a, b\rangle$. Let $\boldsymbol{W}$ denote a certain set of systems $[x, \xi, \delta]$, $\delta > 0$, where $x \in \mathcal{K}$. The following conditions for $\boldsymbol{W}$ are of importance throughout:

I. If $[x, \xi, \delta] \in \boldsymbol{W}$, then $[x, \xi, \delta_1] \in \boldsymbol{W}$ for $\delta_1 < \delta$.

IIa. If $[x, \xi_n, \delta] \in \boldsymbol{W}$, $\xi_n \to \xi_0$, then $[x, \xi_0, \delta] \in \boldsymbol{W}$.

IIb. If $[x_n, \xi, \delta] \in \boldsymbol{W}$ and $x_n(t) \to x(t)$ uniformly in $\langle a, b\rangle$, $x \in \mathcal{K}$, then $[x, \xi, \delta] \in \boldsymbol{W}$.

III. If for every $\xi \in (\alpha, \beta) \subset \langle a, b\rangle, [x_n, \xi, \delta] \in \boldsymbol{W}$, then there exists a subsequence x_{n_i} such that $x_{n_i}(t)$ has a limit (finite or not) almost everywhere in (α, β).

The conditions IIa, IIb follow from

II. If $[x_n, \xi_n, \delta] \in \boldsymbol{W}$, $\xi_n \to \xi_0$, $x_n(t) \to x(t)$ uniformly in $\langle a, b\rangle$, $x \in \mathcal{K}$, then $[x, \xi_0, \delta] \in \boldsymbol{W}$.

In order to be more suggestive we shall use occasionally for the relation $[x, \xi, \delta] \in \boldsymbol{W}$ the following phrase: the function $x \in \mathcal{K}$ has in the neighbourhood $\Delta(\xi, \delta) = (\xi-\delta, \xi+\delta) \wedge \langle a, b\rangle$ of ξ the property $\boldsymbol{W}$. "Property $\boldsymbol{W}$" means "set $\boldsymbol{W}$". We shall now give some examples of the properties $\boldsymbol{W}$ satisfying the above mentioned conditions.

A. Let $\omega(u)$ be continuous and non-decreasing for $u \geqslant 0$, $\omega(u) > 0$ for $u > 0$. The property $\boldsymbol{W}$ is defined by:

$$|x(t)-x(\xi)| \leqslant k\omega(|t-\xi|) \quad \text{for} \quad t \in \Delta(\xi, \delta),$$

where k is a universal constant.

Condition I is obviously satisfied. In order to prove II, let us notice that if $t \in \Delta(\xi_0, \delta)$ and if it is an interior point, then $t \in \Delta(\xi_n, \delta)$ for n sufficiently large, whence

$$|x_n(t)-x_n(\xi_n)| \leqslant k\omega(|t-\xi_n|).$$

It follows by uniform convergence $x_n \to x$ that also $|x(t)-x(\xi_0)| \leqslant k\omega(|t-\xi_0|)$. For the eventual boundary values of t the above inequality holds by continuity. In order to prove III, let us remark that if for $\xi \in (\alpha, \beta)$ the function x_n has the property $\boldsymbol{W}$ in $\Delta(\xi, \delta)$ then, for arbitrary $\xi_1, \xi_2 \in (\alpha, \beta)$ with $|\xi_1-\xi_2| < \delta$, it follows $\xi_2 \in \Delta(\xi_1, \delta)$, i.e. $|x_n(\xi_1)-x_n(\xi_2)| \leqslant k\omega(|\xi_1-\xi_2|)$, and it suffices to apply the theorem of Arzéla.

B. Given $h_1 > h_2 > \ldots, h_n > 0, h_n \to 0$, $\boldsymbol{W}$ means: x belongs to $\mathcal{K}$ if $\xi = b$, $|x(t_n)-x(\xi)| \leqslant k(t_n-\xi)$ for $t_n-\xi = h_n, t_n \in \Delta(\xi, \delta)$ and $t_n \neq b$, if $a \leqslant \xi < b$.

Condition I is self-evident. In order to prove II, suppose $a \leqslant \xi_0 < b$, $t_l = \xi_0+h_l \in \Delta(\xi_0, \delta)$, $t_l \neq b$, $t_l^{(n)} = \xi_n+t_l-\xi_0$. Since $t_l^{(n)} \to t_l$, $t_l^{(n)} \in \Delta(\xi_n, \delta)$ for large n, it follows $|x_n(t_l^{(n)})-x_n(\xi_n)| \leqslant k(t_l^{(n)}-\xi_n)$, whence $|x(t_l)-x(\xi_0)| \leqslant k(t_l-\xi_0)$. In order to prove III, choose $0 < h < \delta$, and then, with given h_l, a nonnegative integer s_l such that $s_l h_l \leqslant h < (s_l+1)h_l$. For l sufficiently large we have $h_l < \delta$ and, provided $\xi+h \leqslant \beta, \xi \geqslant \alpha$, $|x_n(\xi+s_l h_l)-x_n(\xi)| \leqslant |x_n(\xi+s_l h_l)-x_n(\xi+(s_l-1)h_l)|+|x_n(\xi+(s_l-1)h_l)-x_n(\xi+(s_l-2)h_l)|+\ldots+|x_n(\xi+2h_l)-x_n(\xi+h_l)|+|x_n(\xi+h_l)-x_n(\xi)| \leqslant k s_l h_l$.

Hence, since $s_l h_l \to h$ as $l \to \infty$, we obtain $|x_n(\xi+h)-x_n(\xi)| \leqslant kh$, i.e. x_n are equi-continuous in (α, β) and it suffices to apply the theorem of Arzéla.

C. Let $\varphi(u)$ be a function with the same properties as $\omega(u)$ in A. Let $V_\varphi(x, (\alpha, \beta)) = \sup \sum_{\nu=1}^{n} \varphi(|x(t_\nu)-x(t_{\nu-1})|)$, where the upper bound is

taken with respect to all partitions $a = t_0 < t_1 < \ldots < t_n = \beta$ (*φ-variation* of $x(t)$ in $\langle a, \beta\rangle$).

W means: $V_\varphi(x, \Delta(\xi, \delta)) \leqslant k$, k being some universal constant. Condition I is clearly satisfied. In order to prove II, let us consider an arbitrary partition $c = t_0 < t_1 < \ldots < t_m = d$ of the neighbourhood $\bar{\Delta}(\xi_0, \delta) = \langle c, d\rangle$ which is the closure of $\Delta(\xi_0, \delta)$. Choose t', t'' in such a way that $t_0 < t' < t_1$, $t_{m-1} < t'' < t_m$. For n sufficiently large the points of the partition $t', t_1, \ldots, t_{m-1}, t''$ will lie in $\Delta(\xi_n, \delta)$, whence

$$\sum_{\nu=2}^{m-1} \varphi\big(|x_n(t_\nu) - x_n(t_{\nu-1})|\big) + \varphi\big(|x_n(t_1) - x_n(t')|\big) + \varphi\big(|x_n(t_m) - x_n(t'')|\big) \leqslant k$$

whence, again choosing $t' \to t_0$, $t'' \to t_m$, we get for $n \to \infty$

$$\sum_{\nu=1}^{m} \varphi\big(|x(t_\nu) - x(t_{\nu-1})|\big) \leqslant k\,, \quad \text{i.e.} \quad V_\varphi(x, \bar{\Delta}(\xi_0, \delta)) \leqslant k\,.$$

In order to prove III, we note that, if $[x_n, \xi, \delta] \in \boldsymbol{W}$ for every $\xi \in (a, \beta)$, then there exists a constant $r > 0$ independent of n, such that $V_\varphi(x_n, (a, \beta)) \leqslant r$ for $n = 1, 2, \ldots$ Condition III follows now by the generalized theorem of Helly on extracting a convergent subsequence from a family of functions with uniformly bounded φ-variation [2].

2. Let us denote by H^1 the Banach space consisting of the sequences $a = \{a_n\}, \sum_{\nu=1}^{\infty} |a_\nu| < \infty$, where the operations have been defined in a natural manner and norm $\|a\| = \sum_{\nu=1}^{\infty} |a_\nu|$.

THEOREM 1. *Let $\varphi_n(t)$ be continuous and $|\varphi_n(t)| \leqslant K$ in $\langle a, b\rangle$ for $n = 1, 2, \ldots$ Let further*

$$(*) \qquad \varliminf_{m\to\infty} \Big(\varliminf_{n\to\infty} \int_a^\beta (\varphi_n(t) - \varphi_m(t))^2\Big) dt > 0$$

for arbitrary $(a, \beta) \subset \langle a, b\rangle$.

Denote by $\mathcal{K}$ the class of those continuous functions x_a which are of the form

$$x_a = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu(t)\,, \quad \textit{where} \quad a \in H^1\,,$$

and by W *the set of systems $[x, \xi, \delta]$, $x \in \mathcal{K}$, satisfying* I-III. *Write further*

$$A_a = \{\xi\colon\ \xi \in \langle a, b\rangle, [x_a, \xi, \delta] \notin \boldsymbol{W} \text{ for any } \delta > 0\}\,.$$

Under the above assumptions there exists in H^1 a residual set H^* such that, for every $a \in H^*$, A_a is residual in $\langle a, b\rangle$.

Write for brevity $F_n(\alpha,\beta)=\left\{a\colon \left[x_a,\xi,\frac{1}{n}\right]\epsilon\, \boldsymbol{W} \text{ for every } \xi\epsilon(\alpha,\beta)\right\}$. From IIb it follows that $F_n(\alpha,\beta)$ is a closed set in H^1, since if $\|a^k-a\|\to 0$, then x_{a^k} tends to x_a uniformly in $\langle a,b\rangle$.

Suppose that $F_n(\alpha,\beta)$ contains a ball $K(a^0,\varrho)=\{a\colon\|a-a^0\|<\varrho\}$. We can take that $a_n^0=0$ for $n>m$. All the sequences which are of the form $(a_1^0,\ldots,a_m^0,0,0,\ldots,\varrho/2,0,\ldots)$ belong to $K(a^0,\varrho)$, whence if

$$x_k(t)=\sum_{\nu=1}^{m}a_\nu^0\varphi_\nu(t)+\frac{\varrho}{2}\varphi_{m+k}(t)\,,\quad \text{for}\quad k=1,2,\ldots$$

then $\left[x_k,\xi,\frac{1}{n}\right]\epsilon\,\boldsymbol{W}$ for $\xi\epsilon(\alpha,\beta)$. In view of III there exists a subsequence $x_{k_i}(t)$ almost everywhere convergent in (α,β), and the contradiction with $(*)$ follows. We have shown thus that $F_n(\alpha,\beta)$ is a nowhere dense set. Notice further that by IIa the set of those ξ for which, with a given $a\,\epsilon\, H^1$, $\left[x_a,\xi,\frac{1}{n}\right]\epsilon\,\boldsymbol{W}$ is closed, i.e. $E_a^n=\left\{\xi\colon\left[x_a,\xi,\frac{1}{n}\right]\epsilon\,\boldsymbol{W}\right\}$ is a nowhere dense set in $\langle a,b\rangle$ if $n=1,2,\ldots$ for $a\,\epsilon\, H_n=H^1-\bigcup\limits_{\alpha_i<\beta_i}F_n(\alpha_i,\beta_i)$, where $\alpha_i<\beta_i$ are arbitrary rational numbers from $\langle a,b\rangle$.

The set $H^*=\bigcap\limits_{n=1}^{\infty}H_n$ is obviously residual in H^1 as well as $A_a=\langle a,b\rangle-\bigcup\limits_{\nu=1}^{\infty}E_a^\nu$ is for $a\,\epsilon\, H^*$ residual in $\langle a,b\rangle$. Moreover, it follows in virtue of I that if $\xi\,\epsilon\, A_a$, then $[x_a,\xi,\delta]\,\bar\epsilon\,\boldsymbol{W}$ with any $\delta>0$.

THEOREM 2. *Suppose $\boldsymbol{W}$ to satisfy* II-III *and assume in place of* I *a stronger condition* I'; *if $[x,\xi,\delta]\,\epsilon\,\boldsymbol{W}$ and $\Delta(\xi',\delta')\subset\Delta(\xi,\delta)$, then $[x,\xi',\delta']\,\epsilon\,\boldsymbol{W}$. Let $\varphi_n(t)$ denote the functions as in Theorem* 1 *and* $(*)$ *be satisfied for $\langle\alpha,\beta\rangle=\langle a,b\rangle$.*

Under the above assumptions there exists a $\xi_0\,\epsilon\,\langle a,b\rangle$ and a residual set H^* in H^1 such that for $a\,\epsilon\, H^*$, $[x_a,\xi_0,\delta]\,\bar\epsilon\,\boldsymbol{W}$ for any $\delta>0$ ξ_0 can be chosen independently of $\boldsymbol{W}$.

Let $S\Delta$ denote the following property of a sequence $f_n(t)$ of functions defined in the interval Δ: there does not exist any subsequence $f_{n_i}(t)$ almost everywhere convergent in Δ. From $(*)$ taken with $\langle\alpha,\beta\rangle=\langle a,b\rangle$ it follows that $\varphi_n(t)$ has the property $S\langle\alpha,\beta\rangle$. Hence, there exists a subsequence $\varphi_{n^{(1)}}(t)$ having the property $S\langle\alpha_1,\beta_1\rangle$, where $\langle\alpha_1,\beta_1\rangle$ denote $\left\langle a,\frac{a+b}{2}\right\rangle$ or $\left\langle\frac{a+b}{2},b\right\rangle$. In turn, there exists a subsequence $\varphi_{n^{(2)}}(t)$ of the sequence $\varphi_{n^{(1)}}(t)$ having the property $S\langle\alpha_2,\beta_2\rangle$, $\langle\alpha_2,\beta_2\rangle$ being $\left\langle\alpha_1,\frac{\alpha_1+\beta_1}{2}\right\rangle$ or $\left\langle\frac{\alpha_1+\beta_1}{2},\beta_1\right\rangle$ etc. We thus obtain a decreasing sequence

of intervals $\langle a_k, \beta_k\rangle$ with lengths tending to zero such that $\varphi_{n^{(k)}}(t)$ has the property $S\langle a_k, \beta_k\rangle$. The sequence $\{\varphi_{n_i}(t)\}$: $\varphi_{1^{(1)}}(t), \varphi_{2^{(2)}}(t), \ldots, \varphi_{n^{(n)}}(t), \ldots$ has the property $S\langle a_k, \beta_k\rangle$ for $k = 1, 2, \ldots$ Let us denote by ξ_0 the point belonging to $\langle a_k, \beta_k\rangle$ for $k = 1, 2, \ldots$

Obviously, in every neighbourhood $\Delta(\xi_0, \delta)$ $\varphi_{n_i}(t)$ have the property $S\Delta(\xi_0, \delta)$. Let $F_n(a, \beta)$ denote the same set as in the proof of Theorem 1. An analogous argument furnishes then that $F_n\Delta(\xi_0, \delta)$ is a nowhere dense set, $\delta > 0$ being arbitrary. Whence, $H^* = H^1 - \bigcup_{m,n=1}^{\infty} F_n\Delta\left(\xi_0, \frac{1}{m}\right)$ is residual and, for $a \in H^*$, $[x_a, \xi_0, \delta] \notin \boldsymbol{W}$ with any δ, since it follows from $[x_a, \xi_0, \delta] \in \boldsymbol{W}$ by I′ that, for ξ', δ' with $\Delta(\xi', \delta') \subset \Delta(\xi_0, \delta)$, $[x_a, \xi', \delta'] \in \boldsymbol{W}$ contrary to the definition of H^*.

3. Lemma 1. *Let R_1, R_2 be separable metric complete spaces. Let, further, A be a subset of $R_1\times R_2$ with the following properties:* 1) *A satisfies Baire's condition,* 2) *there exists a residual set $R_1^* \subset R_1$ such that the set $\{r_2: (r_1, r_2) \in A\}$ is residual in R_2 if $r_1 \in R_1^*$.*

Under the above assumptions there exists a residual set $R_2^* \subset R_2$ such that the set $\{r: (r_1, r_2) \in A\}$ is residual in R_1 if $r_2 \in R_2^*$.

The proof is based on a theorem from [1] and on the fact that if A is a set of 2-nd category then it must be residual in a certain ball in $R_1\times R_2$.

Lemma 2. *Let $f(t)$ be a continuous function with the period l, $g(t)$ an integrable one in $\langle a, \beta\rangle$, ω_n — a sequence of numbers tending to infinity. Then*

$$\lim \int_a^\beta f(\omega_n t) g(t)\,dt = \frac{1}{l}\int_0^l f(t)\,dt \int_a^\beta g(t)\,dt\,.$$

This lemma is known.

Theorem 3. *Let $f(t)$ be a non-constant in $\langle 0, l\rangle$ continuous function with the period* 1 *and let $\omega_n \to \infty$. Let $\boldsymbol{W}_n$, $n = 1, 2, \ldots$ denote properties subject to conditions I, III. Finally, let A_a denote the set of those $t \in \langle a, b\rangle$ for which the function*

$$(+)\qquad x_a(t) = \sum_{\nu=1}^{\infty} a_\nu f(\omega_\nu t)$$

fails to satisfy $\boldsymbol{W}_n$ $(n = 1, 2, \ldots)$ in each neighbourhood of t.

a) *If, in addition to I, III, $\boldsymbol{W}_n$ satisfy IIa, IIb, then there exists a residual set $H^* \subset H^1$ such that A_a is residual in $\langle a, b\rangle$ for $a \in H^*$;*

b) *If $\boldsymbol{W}_n$ satisfy additionally II, then the set H^* resulting after* a) *can be chosen in such a manner that $\bigcap_{a\in H^*} A_a$ contains a denumerable set, dense everywhere in $\langle a, b\rangle$.*

Let $\varphi_n(t) = f(\omega_n t)$. We shall show that $\varphi_n(t)$ satisfy (*) of Theorem 1. Lemma 1 gives

$$\lim_{n\to\infty} \int_a^\beta (f(\omega_n t) - f(\omega_m t))^2\,dt = \frac{2}{l}\int_0^l f^2(t)\,dt(\beta - a) - \frac{2}{l}\int_0^l f(t)\,dt \int_a^\beta f(\omega_m)\,dt,$$

whence

$$\lim_{m\to\infty}\Big(\lim_{n\to\infty}\int_a^\beta (f(\omega_n t) - f(\omega t_m))^2 dt\Big) = g\,,$$

where

$$g = \frac{2}{l}(\beta - a)\Bigg(\int_0^l f^2(t)\,dt - \frac{1}{l}\Big(\int_0^l f(t)\,dt\Big)^2\Bigg).$$

We proceed by the Schwarz inequality

$$\Big(\int_0^l f(t)\,dt\Big)^2 \leqslant l\int_0^l f^2(t)\,dt\,.$$

The sign $=$ does take place, since otherwise it would be $f(t) = c$ in $\langle a, b\rangle$, contrary to our assumption. We have thus shown that $g > 0$, and the first part of Theorem 3 follows easily from Theorem 1 with $\varphi_n(t) = f(\omega_n t)$. In order to prove b) let us denote by M_n the set of those pairs $(a, \xi) \in H^1 \times \langle a, b\rangle$ for which $\left[x_a, \xi, \frac{1}{n}\right] \in \boldsymbol{W}$, and by M the set of those (a, ξ) for which $[x_a, \xi, \delta] \in \boldsymbol{W}$ with a certain $\delta > 0$. It follows from II that M_n is closed and from I that $M = \bigcup_\nu M_\nu$, whence $N = H^1 \times \langle a, b\rangle - M$ is a Borel-set. From a) and Lemma 1 there follows the existence of a residual set $A \subset \langle a, b\rangle$ such that for $\xi \in A$ in a residual $H^*_\xi \subset H^1$ none of $\boldsymbol{W}_n$ is satisfied in any neighbourhood. Choosing $\xi_1, \xi_2, \ldots$ from A everywhere densely in $\langle a, b\rangle$ we put $H^* = \bigcap_\nu H^*_{\xi_\nu}$.

4. Let now $\mathcal{K}$ denote the set of 2π-periodic, continuous functions. Given for $x \in \mathcal{K}$ a sequence $\boldsymbol{W}_n$ subject to I-III, let us assume that the local property V of the functions from K satisfies the following conditions.

1⁰. x has the property V in ξ if and only if one of the $\boldsymbol{W}_n$'s is satisfied for x in some $\Delta(\xi, \delta)$.

2⁰. If x has the property V in a certain ξ, then the Fourier series of x will be convergent at ξ.

We shall call such a property V *an elementary convergency-test* (in abbreviation *ECT*) and we shall say that x satisfies the test V at the point ξ.

The following convergency-tests for Fourier series are *ECT*:

a) $\overline{\lim\limits_{t\to\xi}}(|x(t) - x(\xi)|)/|t-\xi|^a < \infty\,,\ 0 < a \leqslant 1\,.$

b) $x(t)$ has in a certain neighbourhood $\Delta(\xi, \delta)$ a finite φ-variation that is to say $V_\varphi(x, \Delta(\xi, \delta)) < \infty$ (φ-variation defined as in 1, B).

On choosing φ particularly we get

α) $\varphi(u) = |u|$ Jordan's test

β) $\varphi(u) = |u|^p$, $p > 1$ Wiener's test [4],

γ) $\varphi(0) = 0, \varphi(u) = \exp(-u^{-a})$, for $u > 0$, where $0 < a < \frac{1}{2}$, Young's test [5], [6].

Putting $f(t) = \text{const}$ we obtain as an immediate consequence of Theorem 3:

THEOREM 4. *Let* V_n *be a sequence of ECT. There exists in* $\langle 0, 2\pi\rangle$ *a set* $\xi_1, \xi_2, \ldots, \xi_i, \ldots$ *everywhere dense and an even continuous function* x, *with absolutely convergent Fourier series, such that* V_n $(n = 1, 2, \ldots)$ *do not hold at any* ξ_i.

As in the case of a trigonometric system, we can define analogously the elementary convergency-test for Fourier series with respect to an arbitrary orthonormal system. Yet, in the general case we have to make further assumptions concerning the $\boldsymbol{W}_{ns}$ characterizing our test, viz. in place of I they should satisfy the following condition $(n = 1, 2, \ldots)$:

$$\text{if} \quad [x, \xi, \delta] \in \boldsymbol{W}_n \quad \text{and} \quad \Delta(\xi', \delta') \subset \Delta(\xi, \delta), \quad \text{then} \quad [x, \xi', \delta'] \in \boldsymbol{W}_n .$$

Let us call test of this sort *weak ECT*. E. g. the test of the type $\mathrm{V}_\varphi(x, \Delta(\xi, \delta)) < \infty$ is such a test. We have now the following

THEOREM 5. *Let* $\varphi_n(t)$ *be an orthonormal system in* $\langle a, b\rangle$, $\varphi_n(t)$ *being continuous and satisfying* $|\varphi_n(t)| \leqslant K$ *for* $t \in \langle a, b\rangle$ $n = 1, 2, \ldots$ *Let* V_n *be a sequence of weak ECT. There exists a* ξ_0 *(independent of the test) and a continuous function* x, *with absolutely convergent Fourier series with respect to* $\{\varphi_n(t)\}$, *such that* V_n $(n = 1, 2, \ldots)$ *do not hold at* ξ_0.

The orthonormality of φ_n implies the fulfilment of $(*)$ from Theorem 1, for $\langle \alpha, \beta\rangle = \langle a, b\rangle$. It thus suffices to apply Theorem 2.

Another application of Theorem 2 is as follows:

THEOREM 6. *Let* h_n *be a sequence decreasing to zero. Let* $f(t)$ *have the same properties as in Theorem 3. There exists a residual set* $H^* \subset H^1$ *and a set* $\xi_1, \xi_2, \ldots$ *everywhere dense in* $\langle a, b\rangle$, *such that we have* for $a \in H^*$

$$(\circ) \qquad \overline{\lim_{n\to\infty}} \frac{|x_a(\xi_i + h_n) - x_a(\xi_i)|}{h_n} = \infty \quad \textit{with} \quad i = 1, 2, \ldots,$$

where x_a *is defined by the formula* $(+)$.

In connection with the above theorem let us notice that the existence of functions of the form $(+)$ having the property (o) can be verified under weaker assumptions than those of coefficients $\sum_n |a_n| < \infty$ in the case of $f'(t)$ existing almost everywhere and $\neq 0$ in a set of positive and integrable measure [3]. Hence, Theorem 6 is of interest when $f'(t) = 0$ almost everywhere.

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MATHEMATICS

Linear Differential Equations with Distributions as Coefficients

by

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1. Let I be a compact interval, α a real number, $\langle\alpha\rangle$ the whole part of α, $\{\alpha\} = \alpha - \langle\alpha\rangle$. For $0 \leqslant \alpha < 1$ let $B(\alpha) = B(\alpha, I)$ be the set of such functions $x = x(t)$ on I, that $\int\limits_I |x|^{1/(1-\alpha)} dt < \infty$, $\|x\|_\alpha = \|x\|_{\alpha,I} = \big[\int\limits_I |x|^{1/(1-\alpha)} dt\big]^{1-\alpha}$. For $\alpha \geqslant 1$ let $B(\alpha)$ be the set of such x that $x^{(\langle\alpha\rangle)} \in B(\{\alpha\})$ ($x^{(n)}$ is the n-th derivative of x), $\|x\|_\alpha = \max\limits_{t \in I}[|x(t)|, |x^{(1)}(t)|, \ldots |x^{(\langle\alpha\rangle-1)}(t)|, \|x^{(\langle\alpha\rangle)}\|_{\{\alpha\}}]$. For $\alpha < 0$ let $B(\alpha)$ be the set of such distributions x, that there exists a $y \in B(\{\alpha\})$, $y^{(-\langle\alpha\rangle)} = x$, $\|x\|_\alpha = \inf\limits_y \|y\|_{\{\alpha\}}$. Let $B(\theta)$ contain only the identically zero function. $B(\alpha)$ are obviously Banach spaces.

In this paper the system

$$\frac{dx_i}{dt} = \sum_{j=1}^{k} a_{ij} x_j \text{ *)} \qquad i = 1, 2, \ldots, k \tag{1}$$

will be examined, where $a_{ij} \in B(\alpha_{ij})$, α_{ij} is a real number or $\alpha_{ij} = \theta$.

If α, β are real numbers, let $\alpha > \beta$ have the usual meaning and let us define $\theta > \alpha$ for all real α. If $\alpha + \beta \geqslant 1$, $\max(\alpha, \beta) \geqslant 1$, put $\alpha \circ \beta = \min(\alpha, \beta)$. If $\alpha + \beta \geqslant 1$, $\max(\alpha, \beta) < 1$, put $\alpha \circ \beta = \alpha + \beta - 1$. Let $\alpha \circ \theta = \theta \circ \alpha = \theta$ for all real α. Obviously: if $a \in B(\alpha)$, $b \in B(\beta)$, $\alpha + \beta \geqslant 1$, then $ab \in B(\alpha \circ \beta)$.

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*) Suppose that $a^{(n)} \in B(0)$, $f = F^{(n)}$, $F \in B(0)$ and $a^{(n)}F \in B(0)$ (for an appropriate $n = 0, 1, 2, \ldots$). Then the product af is defined by the formula

$$af = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (a^{(j)}F)^{(n-j)}$$

(cf. [3], par 4).

2. Troughout this paper it will be assumed that $a_{ij} \in B(\alpha_{ij})$, $\min_j \alpha_{ij} < 0$ for $i = 1, 2, \ldots, k$ and that the system

$$(2) \qquad \xi_i - 1 = \min_j (\alpha_{ij} \circ \xi_j)$$

has a solution, which will be denoted by $\xi_i x$ (The system (2) has at most one solution).

THEOREM 1. *If the distributions x_i fulfil* (1), *then $x_i \in B(\xi_i)$, $i = 1, 2, \ldots, k$.*

In order to establish the existence theorem the initial conditions must be defined in an appropriate way, as it may happen that $\xi_i < 1$ for some i. Let $\Phi(\xi)$ for $\xi \geqslant 1$ be the set of linear functionals on the space of continuous functions on I with the usual norm and let $\Phi(\xi)$ for $\xi < 1$ be the set of linear functionals φ on $B(\xi)$ with the norm

$$\|\varphi\|_\xi = |I|^{1-\xi} \sup_{\|x\|_\xi \leqslant 1} |\varphi x| \; . \; ^{*)}$$

THEOREM 2. *If $a > 0$ and $K > 0$, then there exist such $\sigma > 0$ and $\varepsilon > 0$* **) *that the following assertion holds*:

Let $|I| \leqslant \sigma$, $\varphi_i \in \Phi(\xi_i)$, $\|\varphi_i\|_{\xi_i} \leqslant K$ $(i = 1, 2, \ldots, k)$ $\varphi_i x = 1$ if $x \equiv 1$ on I, $\|a_{ij}\|_{\alpha_{ij}} < \varepsilon$ if $\alpha_{ij} \leqslant 0$,

$$(3) \qquad \|a_{ij}\|_{\alpha_{ij}} \leqslant a \quad \text{if} \quad \alpha_{ij} > 0 \, .$$

Let λ_i be real numbers. Then there exists such a solution x_i of (1) *that*

$$\varphi_i x_i = \lambda_i \, , \quad i = 1, 2, \ldots k \, .$$

This solution is determined uniquely.

Note 1. Let $\xi < 1$, $l = -\langle \xi \rangle + 1$ and put

$$P_l(z(t)) = \gamma_{0l} z(0) + \gamma_{1l} z(1/l) + \ldots + \gamma_{ll} z(1) \, ,$$

where the coefficients γ_{ij} are determined by the following conditions: $P_l(z(t)) = 0$ if $z(t) = t^r$, $z = 0, 1, 2, \ldots l-1$, $P_l(t^l) = l!$ If $x \in B(\xi)$, $y^{(l)} = x$, $\langle t_0, t_0 + h \rangle \subset I$, put $\psi x = \frac{1}{h^l} P_l(y(t_0 + ht))$. Then $\psi \in \Phi(\xi)$, $\psi x = 1$ if $x \equiv 1$ on $\langle t_0, t_0 + h \rangle$ and

$$(4) \qquad \|\psi\|_\xi \leqslant C_l (|I|/h)^l \, .$$

Note 2. Let $a_{ij} \in B(\alpha_{ij}, I)$. Let a fulfil (3) and take K great enough (for example $K > 1$, $K > C_{l_i} 4^{l^i}$ if $\xi_i < 1$; l_i, C_i were introduced in Note 1). Let us find ε and σ according to Theorem 2. Obviously, there exists such a δ, $0 < \delta \leqslant \sigma$ that $\|a_{ij}\|_{\alpha_{ij}, I_1} \leqslant \varepsilon$ for $\alpha_{ij} \leqslant 0$ if $I_1 = \langle a, b \rangle \subset I$, $|I_1| \leqslant \delta$.

*) $|I|$ is the length of I.

**) σ and ε depend on a, K and α_{ij}.

If $\xi_i \geqslant 1$, put $\varphi_i x_i = x_i(a)$; if $\xi_i < 1$, put $\varphi_i x_i = \psi_i x_i$, where ψ_i is defined in Note 1 ($t_0 = a, h = |I_1|/4$, $l_i = -\langle \xi_i \rangle + 1$. Theorem 2 may be used on I_1. In a similar way the solution x_i may be continued from I_1 to $\langle b - |I_1|/4, b + 3|I_1|/4\rangle$ and after a finite number of steps the solution x_i may be continued to I.

3. Let $a = a(t)$ be a distribution on I, $t_0 \in I$, $a < 1$. Let us say that a assumes the value c at t_0 with respect to $B(a)$, if $a \in B(a, I)$ and if $\|a(t_0 + \lambda t) - y(t)\|_a \to 0$ with $\lambda \to 0$, where $y(t) = c$ on I; in this case we write $[a, t_0]_a = c$. If $a \geqslant 1$, $a \in B(a)$, put $[a, t_0]_a = a(t_0)$. If $[a, t_0]_a = c$, then $[a, t_0]_\beta = c$ for $\beta < a$ and a has the value c at t_0 in the sense of Łojasiewicz ([1], § 16).

Theorem 3. *Let the values* $[a_{ij}^{(-1)}, t_0]_{a_{ij}+1}$ *exist* *). *Then, for every solution* x_i *of* (1), *the values* $[x_i, t_0]_{\xi_i}$ *exist. For arbitrary real* λ_i *there exists a unique solution* x_i *of* (1), $[x_i, t_0]_{\xi_i} = \lambda_i$.

4. Let a_i, $i = 1, 2, \dots k$ be real numbers or $a_i = \theta$, $\min\limits_i a_i < \theta$ and let ξ be the solution of

$$\xi - k = \min_i a_i \circ (\xi - k + i) .$$

Theorem 4. *If* $a > 0$, *then there exists such* $\sigma > 0$ *and* $\varepsilon > 0$ *that the following assertion holds*:

Let $|I| \leqslant \sigma$, $\|a_i\|_{a_i} \leqslant \varepsilon$ *if* $a_i \leqslant 0$, $\|a_i\|_{a_i} \leqslant a$ *if* $a_i > 0$, $t_1, t_2, \dots t_k \in I$, $t_1 < t_2 < \dots < t_k$.

Then for arbitrary real numbers λ_i *there exists a solution* x *of*

$$x^{(k)} + a_1 x^{(k-1)} + \dots + a_k x = 0 ,$$

$x(t_i) = \lambda_i$, $i = 1, 2, \dots k$.

The solution x *is determined uniquely.*

Theorem 4 is deduced from Theorem 2 and has a close relation to the theorem of de la Valée Poussin ([2], Chap. IV, par. 1).

Note 3. In the equation

$$x^{(k)} + a x = 0 \tag{5}$$

we may admit $a \in B\left(\dfrac{-k+1}{2}\right)$. $\Bigl($In order to use Theorem 2 we transform Eq. (5) to a system in the usual way and put $a_{k1} = \dfrac{-k+1}{2}$, $a_{i,i+1} = \dfrac{k+1}{2}$, $i = 1, 2, \dots, k-1$ and $a_{ij} = \theta$ for the remaining $(ij)\Bigr)$. Especially we may take $a = |t|^\beta$, $\beta > \dfrac{-k+1}{2}$, $I = \langle -1, 1\rangle$ or $a = g(t)\,|t|^\beta$, where

*) $a^{(-1)}$ is a primitive distribution to a.

is measurable and bounded. According to Theorem 4, there exist σ and ε with the following property:

Let $t_1 < t_2 < \ldots t_k$, $\langle t_1, t_k \rangle = I_1$, $|I_1| \leqslant \sigma$. Let there exist such a function A that $A^{(\varrho)} = a$, $\varrho = -\left\langle\dfrac{-k+1}{2}\right\rangle$ and $\|A\|_{\eta, I_1} \leqslant \varepsilon$, $\eta = \left\{\dfrac{-k+1}{2}\right\}$.

Then for arbitrary real λ_i there exists a unique solution x of (5), $x(t_i) = \lambda_i$, $i = 1, 2, \ldots k$.

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MATHEMATICS

On Random Operator Equations in Banach Spaces

by

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1. Let U be a bounded closed subset in finite dimensional Euclidean space R_k, and let $x(u)$ be a measurable function defined on U. A large number of linear functional equations studied in functional analysis can be written in the form

$$(T-\lambda I)x(u) = y(u)\,, \tag{1}$$

where x and y are elements of a concrete Banach space $\mathfrak{X}$, T is a linear, not necessarily bounded, operator with domain and range in $\mathfrak{X}$; I the identity operator, and λ an arbitrary constant. It is well known that the class of all linear bounded operators (endomorphisms) on $\mathfrak{X}$, which we denote by $\mathcal{B}(\mathfrak{X})$, is a Banach algebra [4].

In this note we wish to begin the study of the stochastic (or probabilistic) analogue of Eq. (1); that is, the random operator equation

$$(T(\omega)-\lambda I)x(u,\omega) = y(u,\omega) \tag{2}$$

which arises in the study of certain linear functional equations in probabilistic functional analysis; especially in the theory of stochastic boundary value problems. In Eq. (2) $x(u,\omega)$ (also $y(u,\omega)$) is a generalised random variable with values in a Banach space $\mathfrak{X}$; that is, $x(u,\omega)$ is a μ-measurable mapping of the product space $U\times\Omega$ (where $(\Omega, \mathcal{A}, \mu)$ is a probability space) into the Banach space $\mathfrak{X}$ such that for every fixed $u\in U$, and each Borel set C,

$$\big\{\{\omega\colon x(u,\omega)\in C\}\colon C\in \mathcal{C}\big\}\subset \mathcal{A};$$

where $\mathcal{C}$ is the σ-algebra of Borel subsets of $\mathfrak{X}$; $T(\omega)$ is a random operator on $\Omega\times\mathfrak{X}$ into $\mathfrak{X}$; that is, T is an operator depending on a random parameter $\omega\in\Omega$, such that

$$\big\{\{\omega\colon T(\omega)x\in C\}\colon x\in\mathfrak{X}\,,\ C\in\mathcal{C}\big\}\subset\mathcal{A},$$

and I is the operator which is identical for every $\omega\in\Omega$.

We shall assume throughout this note that $T(\omega)$ is

(i) linear (i.e. $T(\omega)(\alpha x_1+\beta x_2) = \alpha T(\omega)x_1+\beta T(\omega)x_2$ for every $\omega \in \Omega$, $x_1, x_2 \in \mathfrak{X}$ and $\alpha, \beta \in R$,

(ii) bounded (i.e. there exists a mapping $c(\omega)$ on Ω into R such that for all $\omega \in \Omega$ and $x \in \mathfrak{X}$, $\|T(\omega)x\| \leqslant c(\omega)\|x\|$; and

(iii) measurable (i.e. the inverse image of every measurable set is measurable).

We denote by $\mathscr{B}_\omega(\mathfrak{X})$ the Banach algebra of random operators $T(\omega)$; that is, $\mathscr{B}_\omega(\mathfrak{X})$ is the collection of all measurable random endomorphisms of $\mathfrak{X}$ for which the usual properties of a Banach algebra hold for all $\omega \in \Omega$.

2. The main problem we wish to consider in this note is that of the existence and measurability of the resolvent operator associated with a random operator $T(\omega)$. Let $T_\lambda(\omega) = T(\omega)-\lambda I$. The values of λ for which $T_\lambda(\omega)$ has a bounded inverse, say $R_\lambda(T, \omega)$, with domain dense in a Banach space of generalised random variables form the *ω-resolvent set* $\varrho(T(\omega))$ of $T(\omega)$. The operator $R_\lambda(T, \omega) = T_\lambda^{-1}(G)$ will be called the *resolvent* of $T(\omega)$, and since it also depends on ω it will be a random operator on $\mathfrak{X}$ into itself, and element of $\mathscr{B}_\omega(\mathfrak{X})$.

We now assume that the Banach space $\mathfrak{X}$ under consideration is the space of all functions $x(u, \omega)$ on $U\times\Omega$ into $\mathfrak{X}$ which for each fixed $u \in U$ are μ-measurable and satisfy the condition given above.

We state the following

THEOREM. 1. *Let $T(\omega)$ be a measurable endomorphism on a separable Banach space $\mathfrak{X}$ of generalised random variables, and let the random operator $T_\lambda(\omega)$ be invertible for each ω separately in the subset $\Omega_0 = \{\omega\colon |\lambda| \geqslant \|T(\omega)\|\}$. Then the random resolvent operator $R_\lambda(T, \omega)$ exists for all $\omega \in \Omega_0$, and is a bounded μ_0-measurable random operator on $\mathfrak{X}$, where μ_0 is the measure defined on the σ-algebra $\Omega_0 \cap \mathscr{A}$, with $0 < \mu_0 \leqslant \mu_0^2$.*

The proof of this theorem is based on a classical result concerning the inversion of linear bounded operators, and a result of O. Hanš [3].

We remark that for all $\omega \in \Omega_0$ the spectrum of the random operator $T(\omega)$ is contained in a circle of radius $\|T(\omega)\|$ with center at zero. Hence, the radius of the circle enclosing the spectrum depends on ω, and is given by the random norm $\|T(\omega)\|$.

It is well known (cf. for example [6], [7]) that if S is a bounded linear operator with domain in a Banach space, then all λ such that $|\lambda| \geqslant \|S\|$ belong to the resolvent set $\varrho(S)$, and for these values of λ the resolvent operator $R_\lambda(S)$ admits the series representation

$$R_\lambda(S) = -\sum_{n=1}^{\infty} \lambda^{-n} S^{n-1}. \tag{3}$$

We now give a stochastic analogue of the above result. The proof of the theorem we state is based on results of Hanš [2] and the following

LEMMA. *Let $S_k(T, \omega)$ denote the partial sums*

$$S_k(T, \omega)[x] = \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}(\omega, x) , \tag{4}$$

where

$$T^n(\omega, x) = T(\omega, T^{n-1}(\omega, x)) , \tag{5}$$

with $T^0(\omega, x) = I$, is the n-th iterate of a bounded μ-measurable random endomorphism $T(\omega, x)$ with domain a separable Banach space $\mathfrak{X}$, and λ is an arbitrary, but fixed, complex constant. If the probability measure μ_0 is complete, then the sequence of random operators $S_1, S_2, \ldots$ converges weakly almost surely to a random operator $S \in \mathcal{B}_\omega(\mathfrak{X})$, $\omega \in \Omega_0$ if and only if

(i) $\mu\left(\omega\colon \bigcup_{k=1}^{\infty} \{S_k(T, \omega)\} \text{ is weakly compact}\}\right) = 1$,

and at the same time

(ii) $\mu\left(\bigcap_{x \in \mathfrak{X}} \{\omega\colon \lim_{k\to\infty} x^*(\{S_k(T, \omega) - S(T, \omega)\}[x]) = 0\}\right) = 1$

for every linear functional $x^*(x) \in X$, where X is a subset of the adjoint space $\mathfrak{X}^*$, that is total on the whole space $\mathfrak{X}$.

THEOREM 2. *Let $T(\omega)$ be a measurable endomorphism with domain a separable Banach space $\mathfrak{X}$, and let the partial sums of the iterates of $T(\omega)$, as given by (5), satisfy the conditions of the above lemma. Then all λ such that $|\lambda| \geqslant \|T(\omega)\|$, where $\omega \in \Omega_0$, belong to the ω-resolvent set $\varrho(T(\omega))$, and for these values of λ the random resolvent operator $R_\lambda(T, \omega)$ admits the series representation*

$$R_\lambda(T, \omega) = -\sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}(\omega) , \tag{6}$$

the series converging weakly almost surely in $\mathcal{B}_\omega(\mathfrak{X})$, $\omega \in \Omega_0$.

Finally, we state the following

THEOREM 3. *For all $\lambda \in \varrho(T(\omega))$, and for every μ measurable function $y(u, \omega) \in \mathfrak{X}$, the random operator equation $(T(\omega) - \lambda I)x = y$ has for every $u \in U$ a uniquely determined μ_0-measurable random solution*

$$x(u, \omega) = R_\lambda(T, \omega) y(u, \omega) \in \mathfrak{X} . \tag{7}$$

3. Remarks. (a). We refer to [1] for proofs of the theorems and lemma stated above, and for an application to the theory of random integral equations in Banach spaces. (b). For other studies on the inversion of rnadom transformations we refer to O. Hanš [3] and A. Špaček [5].

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МАТЕМАТИКА

Б. БОЯРСКИЙ

О ПЕРВОЙ КРАЕВОЙ ЗАДАЧЕ ДЛЯ СИСТЕМ УРАВНЕНИЙ ЭЛЛИПТИЧЕСКОГО ТИПА ВТОРОГО ПОРЯДКА НА ПЛОСКОСТИ

Представлено С. МАЗУРОМ 18 июля 1959

Система уравнений

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+Eu_x+Fu_y+Du=g\,, \tag{1}$$

где A, B, C, E, F, D заданные $n\times n$ матрицы-функции, $u=(u_1, \ldots, u_n)$ и $g=(g_1, \ldots, g_n)$ искомый и заданный векторы, называется эллиптической, если

$$\det|A\lambda^2+2B\lambda+C|\neq 0\,, \qquad \det A\neq 0 \tag{2}$$

для всех действительных λ.

Пусть система (1) задана в области D ограниченной гладкой кривой Γ. Первая краевая задача есть задача нахождения решения системы (1), непрерывного в $D+\Gamma$ и удовлетворяющего условию

$$u|_\Gamma=0\,. \tag{3}$$

Как показал А. В. Бицадзе [1], задача (1)-(3) для эллиптических систем, (уже при $n=2$), может допускать бесконечное число линейно независимых решений. Это явление не может иметь места для так называемых сильно эллиптических систем уравнений [8]. Более широкие классы уравнений, чем класс сильно эллиптических систем, для которых задача (1)-(3) корректна, можно выделить на основе работ автора [2], [3] и А. И. Вольперта [9], [10].

В настоящем предварительном сообщении, ограничиваясь случаем $n=2$, дается другой подход к задаче (1)-(2). Указывается также обобщение полученных результатов на системы порядка $m>2$.

1. Гомотопическая классификация систем уравнений эллиптического типа ($n=2$)

Рассмотрим полиномиальную матрицу

$$\mathfrak{A}(\lambda)=A\lambda^2+2B\lambda+C=\big(a_{ij}(\lambda)\big) \qquad i,j=1,2 \tag{4}$$

2-го порядка. Матрица (4) есть характеристическая матрица системы (1). Ставится вопрос об описании компонент связности множества всех полиномиальных матриц вида (4) при условии (2). Это равносильно вопросу о ком-

понентах связности в классе систем 2-го порядка эллиптического типа с двумя независимыми переменными.

В соответствии с (2) возможны два случая

$$(5) \qquad 1)\ \ \det\mathfrak{A}(\lambda) > 0\,, \qquad 2)\ \ \det\mathfrak{A}(\lambda) < 0\,.$$

Пусть имеет место (5.1). Рассмотрим комплексный полином

$$(6) \qquad a(\lambda) = a_{11}(\lambda) + a_{22}(\lambda) + i\big(a_{21}(\lambda) - a_{12}(\lambda)\big)$$

и обозначим через λ_1 и λ_2 корни уравнения $a(\lambda) = 0$. Из (5.1) следует, что λ_1 и λ_2 не лежат на оси $\operatorname{Im}\lambda = 0$. Поэтому имеют место три возможности:

$$(7) \qquad 1)\ \ \operatorname{Im}\lambda_1 > 0\,,\ \ \operatorname{Im}\lambda_2 > 0\,, \qquad 2)\ \ \operatorname{Im}\lambda_1\,,\ \ \operatorname{Im}\lambda_2 < 0\,,$$

$$3)\ \ \operatorname{Im}\lambda_1 < 0,\ \ \operatorname{Im}\lambda_2 < 0\,.$$

Множество систем (1), для которых имеет место неравенство (7.i), при условии (5.1), обозначим соответственно через E_i, $i = 1, 2, 3$.

Теорема 1. *Множество всех систем двух уравнений эллиптического типа второго порядка состоит из* 6-*ти компонент. Они описываются неравенствами* (5.i), (7.k), $i = 1, 2$, $k = 1, 2, 3$.

Если рассматривается система двух уравнений m-того порядка, $m > 2$, характеристическая матрица которой имеет вид

$$(8) \qquad \mathfrak{A}(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \ldots + A_0\,,$$

то полином (6) будет полиномом m-того порядка, допускающим m недействительных корней. Условия вида (7) дают тогда $m+1$ возможностей и вместе с неравенствами (5.i) позволяют доказать теорему 1′.

Теорема 1′. *Множество всех систем двух уравнений эллиптического типа порядка m состоит из* $2(m+1)$ *компонент.*

Замечание 1. В теоремах 1 и 1′ существенно то, что рассматриваются уравнения с действительными коэффициентами. Если допустить комплексные коэффициенты, то все уравнения эллиптического типа образуют лишь одну компоненту.

Примеры. Система $\Delta u_1 = 0$, $\Delta u_2 = 0$ принадлежит к классу E_2. Система А. В. Бицадзе [1]

$$(9) \qquad u_{1xx} - u_{1yy} - 2u_{2xy} = 0\,, \qquad u_{2yy} - u_{2xx} - 2u_{1xy} = 0$$

принадлежит к классу E_3. Аналогичная ей система

$$u_{1xx} - u_{1yy} + 2u_{2xy} = 0\,, \qquad u_{2xx} - u_{2yy} - 2u_{1xy} = 0$$

принадлежит к классу E_1. При $m > 2$ примеры систем различных классов $E_1, \ldots, E_{m+1}$ можно получить подходящим дифференцированием указанных выше примеров.

Замечание 2. Вопрос о классификации полиномиальных матриц (8) можно поставить еще так: когда матрицу (8) в классе действительных полиномиальных матриц (8) при условии (2) можно привести непрерывной деформацией

к диагональному виду и когда такое приведение невозможно? Если такое приведение возможно, то искомые функции u_1 и u_2 в соответствующей системе дифференциальных уравнений могут быть непрерывной деформацией „отделены" друг от друга. Тогда можно сказать, что в исходной системе они „зацеплены несущественно". Если такое отделение невозможно, то скажем, что в исходной системе u_1 и u_2 „зацеплены существенно". В системе уравнений теории упругости зацепление „несущественно" (она принадлежит к классу E_2) при всех допустимых значениях параметров. В системе (9) зацепление существенно. Как увидим в дальнейшем, задача Дирихле (или Неймана) для систем, для которых зацепление несущественно, всегда корректна, т. е. эта задача допускает самое большее конечное число линейно независимых решений и конечное, равное ему, число условий разрешимости (неоднородной задачи).

Следствие. *Сильно эллиптические системы принадлежат к классу систем с „несущественным зацеплением"*.

Замечание 3. Все сказанное относится также к уравнениям с переменными непрерывными коэффициентами. Полином a зависит в этом случае от точки z, $a = a(\lambda; z)$, однако во всех точках области эллиптичности может реализоваться лишь одна и та же из возможностей (7). Поэтому для проверки к какому классу принадлежит данная система, достаточно исследовать корни полинома $a(\lambda; z)$ в какой нибудь одной точке рассматриваемой области.

Замечание 4. Теоремы 1 и 1' являются частичным ответом (при $n = 2$) на вопрос поставленный в совместном докладе И. М. Гельфанда, И. Г. Петровского и Г. Е. Шилова на III Всесоюзном математическом съезде в Москве, в 1956 г. Приведенный там вывод о двух компонентах связности множества эллиптических систем, при $n = 2$ и $m = 2$ сделанный на основе одного результата М. И. Вишика (ср. [8], § 8), не соответствует действительности. Следует отметить, что рассмотренные в ([8], § 8) системы уравнений не являются общим видом системы уравнений 2-го порядка эллиптического типа.

2. Применения к теории первой краевой задачи

В этом пункте мы изучим, при $n = 2$ и $m = 2$, связь вопросов классификации с теорией граничных задач. Ограничение $n = 2$ и $m = 2$ вызвано спецификой этого случая, которая позволяет успешно применять специальные методы разработанные в [5], [6] и [4] применительно к уравнениям 1-го порядка. Наше исследование ограничиваем классом E_2.

Прежде всего запишем систему (1) в комплексной форме, вводя функцию $f = u_1 + iu_2$ и комплексные дифференциальные операторы $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$.

Лемма 1. *Любая система* (1) *класса* E_2 *может быть записана в виде*

$$f_{z\bar z} + \alpha f_{zz} + \beta f_{\bar z\bar z} + \gamma \overline{f}_{zz} + \delta \overline{f}_{\bar z\bar z} + T(f) = g_1\,, \tag{10}$$

где $T(f)$ — *операторы низшего порядка и* $\alpha, \beta, \gamma, \delta$ — *комплексные функции.*

Множество тех систем $(\alpha, \beta, \gamma, \delta)$ комплексных чисел, для которых система (10) эллиптична и принадлежит к классу E_2, обозначим через S. S есть звездное множество относительно точки $(0, 0, 0, 0)$. Пусть $\hat{S}$ полицилиндр

$$|\alpha| + |\beta| + |\gamma| + |\delta| < 1 . \tag{11}$$

Показывается, что $\hat{S} \subset S$. Однако, S более обширно чем $\hat{S}$. Например, система $f_{\bar{z}\bar{z}} - \nu(f_{zz} - f_{\bar{z}\bar{z}}) = 0$ принадлежит к E_2 при любом действительном ν и, очевидно, условие (11) не выполняется, если $|\nu|$ достаточно велико. Однако, можно показать, что с точностью до членов низшего порядка любая система (10) класса E_2 может быть простыми преобразованиями приведена к случаю, когда (11) выполняется.

Отметим прежде всего, что неравенство (11) выполняется в следующих пяти случаях: $\alpha = \beta = 0$, $\alpha = \gamma = 0$, $\alpha = \delta = 0$, $\beta = \gamma = 0$, $\beta = \delta = 0$.

В случае $\gamma = \delta = 0$, который выше не перечисляется и к которому сводятся системы разобранные, при постоянных коэффициентах, М. И. Вишиком [8], приведение к случаю (11) делается посредством замены независимого переменного $\zeta = \zeta(z)$ по квазиконформному отображению, отвечающему системе Бельтрами $\zeta_{\bar{z}} - q(z)\zeta_z = 0$, где $q(z)$ есть корень уравнения

$$\alpha + q + \beta q^2 = 0 \tag{12}$$

такой, что $|q| < 1$. Доказывается, что один из корней уравнения (12) удовлетворяет в классе E_2 неравенству $|q_1| < 1$, а второй — неравенству $|q_2| > 1$. В общем случае, когда $\gamma \neq 0$ или $\delta \neq 0$, к преобразованию независимой переменной необходимо добавить преобразование искомой функции $F = af + b\bar{f}$, (*), где a и b комплексные функции. При этом, однако, в случае переменных коэффициентов, для осуществления такого преобразования глобально, во всей рассматриваемой области, накладываются некоторые условия на систему (1): кроме некоторой гладкости коэффициентов требуется еще возможность выделения однозначной непрерывной ветви корня уравнения (2) во всей рассматриваемой области. Построение преобразования несколько громоздко и подробностями мы пренебрегаем.

Исследование задачи Дирихле для системы (10), удовлетворяющей условию (11), может быть проведено при весьма слабых ограничениях на коэффициенты. Если вместо (11) предположить, что в рассматриваемой области выполняется условие равномерной эллиптичности,

$$|\alpha| + |\beta| + |\gamma| + |\delta| \leqslant \mu_0 < 1 , \quad \mu_0 - \text{const}, \tag{11'}$$

то коэффициенты $\alpha, \beta, \gamma, \delta$ в остальном можно считать любыми измеримыми функциями; о коэффициентах, входящих в оператор $T(f)$, достаточно предположить, что они принадлежат к $L_p(D)$, $p > 2$. Без ограничения общности можно считать, что D есть единичный круг, ибо условие (11′) инвариантно при конформных преобразованиях независимой переменной в системе (10). Представляя тогда искомую функцию $f(z)$ по формуле

$$f(z) = \iint_D G(z, \zeta)\,\varrho(\zeta)\,dD_\zeta ,$$

где $G(z,\zeta)$ — функция Грина однородной задачи Дирихле для уравнения Лапласа в единичном круге, получим в силу (10) для комплексной плотности $\varrho(\zeta)$ эквивалентное интегральное уравнение вида [4]-[7]

$$\varrho(\zeta)+aS_1(\varrho)+\beta S_2(\varrho)+\gamma\overline{S_1(\varrho)}+\delta\overline{S_2(\varrho)}+T_1(\varrho)=g_1 \tag{13}$$

где S_1, S_2 — ограниченные операторы в $L_p(D)$.

Это есть сингулярное двухмерное интегральное уравнение. Рассматривая его в пространстве $L_p(D)$, $2\leqslant p<2+\varepsilon$, при достаточно малом $\varepsilon>0$, нетрудно видеть, что оно эквивалентно уравнению Фредгольма второго рода с вполне непрерывным оператором. В самом деле, согласно результатам Кальдерона и Зигмунда [12], $\|S_1\|_{L_p}\leqslant A_p$, $\|S_2\|_{L_p}\leqslant A_p$, причем $A_p\to 1$ при $p\to 2$, ибо $\|S_1\|_{L_2}=\|S_2\|_{L_2}=1$.

Тогда из условия (11′) следует, что оператор

$$\varrho+aS_1(\varrho)+\beta S_2(\varrho)+\gamma\overline{S_1(\varrho)}+\delta\overline{S_2(\varrho)}=h$$

имеет в $L_p(D)$ ограниченный обратный, $\varrho=R(h)$, $\|R\|_{L_p}\leqslant\dfrac{1}{1-\mu_0A_p}$ и уравнение (13) эквивалентно уравнению

$$\varrho+RT_1(\varrho)=Rg_1=g_2\,,$$

в котором оператор $RT_1(\varrho)$ вполне непрерывен.

Теорема 2. *Задача Дирихле для уравнения* (10), *удовлетворяющего условию* (11′) *приводится к эквивалентному уравнению Фредгольма. В силу указанных преобразований, при наложенных условиях, этот факт имеет место для общих систем* (1) *класса* E_2 (*с несущественным зацеплением*), *если только коэффициенты главной части системы непрерывны и обладают обобщенными первыми производными класса* L_p, $p>2$. *В частности, индекс задачи Дирихле в классе* E_2 *всегда равен нулю.*

Напомним, что индексом граничной задачи называется разность числа решений однородной задачи и числа условий разрешимости неоднородной задачи.

Заметим, что наложенные при глобальном приведении системы (10) к типу (11) условия на корни уравнения (2) и требуемая гладкость коэффициентов главной части системы несущественны при локальной постановке вопроса, в частности, при решении граничной задачи в достаточно малых областях. Эти ограничения в общем случае вызваны применяемым методом и, по всей вероятности, Теорема 2 справедлива для систем с измеримыми коэффициентами главной части, равномерно принадлежащих к классу E_2. Аналогично [7] может быть рассмотрена задача Неймана $\left.\dfrac{\partial f}{\partial n}\right|_{\Gamma}=0$ и для нее, в классе E_2, справедлив аналог Теоремы 2.

Теорема 2 указывает на существенное отличие поведения систем уравнений класса E_2, в отношении теории граничных задач, в сравнении с остальными классами E_1 и E_2. В то время как в E_1 и E_2 возможны явления некорректности основных краевых задач, в E_2 это невозможно. Теория систем

уравнений класса E_2 в этом отношении сохраняет полную аналогию с теорией одного уравнения эллиптического типа. Аналогичный факт наблюдается и в теории первой краевой задачи ($m = 2k$) для систем порядка $m > 2$. При условии аналогичном (5.1) там имеется $m+1$ классов, в каждом из которых за исключением одного, содержащего систему полигармонических уравнений $\Delta^k u_1 = 0$, $\Delta^k u_2 = 0$, можно указать примеры первой краевой задачи, для сколь угодно малых областей, допускающие бесчисленное множество линейно независимых решений.

Не следует однако, думать, что в классах E_1 и E задача Дирихле всегда обладает этим свойством. На основании результатов работы [3] нетрудно построить примеры задач Дирихле в классе E_1 или E_3, решение которых сводится к системам Фредгольма или к системам сингулярных интегральных уравнений нормального типа.

Отметим, наконец, следующий невыясненный вопрос: будет ли задача Дирихле для систем класса E_2 с переменными коэффициентами без младших членов всегда допускать единственное решение? Для уравнений класса $\hat{S}$ это так. но при переходе к этому случаю, вообще говоря, появляются члены низшего порядка, присутствие которых и затрудняет ответ.

ИНСТИТУТ ГИДРОДИНАМИКИ, СИБИРСКОЕ ОТДЕЛЕНИЕ АН СССР

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B. BOJARSKI, ON THE FIRST BOUNDARY VALUE PROBLEM FOR ELLIPTIC SYSTEM OF SECOND ORDER IN THE PLANE

In the present work a homotopic classification of systems of two equations of type (1) is given. Moreover, it is proved, that in one of the components of the system of equations (1) — (5.1), the Dirichlet problem [1]-[3] is equivalent to a system of Fredholm equations.

It is to be noticed that the index in this component equals 0.

Generalizations to systems of order $m > 2$ are also given.

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THEORETICAL PHYSICS

Geometrical Interpretation of Conservation Laws in the Spinor Space

by

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It is well known that of the conservation laws encountered in the theory of elementary particles only the conservation laws of energy-momentum and angular momentum possess a geometrical origin in the properties of space-time homogeneity and isotropy. There is obviously no possibility of interpreting geometrically within the framework of space-time such laws as conservation of charge, baryon number, isotopic spin etc. Therefore, in order to provide a geometrical basis also for these laws, it is necessary to consider other spaces.

In a previous work we introduced the spinor space as a possible means for the description of physical laws (cf. e.g. [1]-[3]). In the present paper the geometrical interpretation of conservation laws in the framework of this space is carried out explicitly.

Let us consider first the direct product $\boldsymbol{c}\cdot\boldsymbol{c}'$ of the unimodular group $\boldsymbol{c}$ connected with ordinary spin and another unimodular group $\boldsymbol{c}'$ connected with isotopic spin. The lowest true representation of $\boldsymbol{c}\cdot\boldsymbol{c}'$ is given by the four independent spaces

$$Z_{\alpha;\beta}, \quad Z_{\dot{\alpha};\dot{\beta}}, \quad Z_{\alpha;\dot{\beta}}, \quad Z_{\dot{\alpha};\beta}, \quad (\alpha, \beta = 1, 2)\,, \tag{1}$$

where the indices before the semicolon transform with respect to $\boldsymbol{c}\cdot\boldsymbol{c}'$ according to $\boldsymbol{c}$, and the indices after the semicolon according to $\boldsymbol{c}'$. The spaces (1) are considered as the geometrical basis for the description of physical laws. To reduce the number of dimensions we introduce the reality conditions

$$Z_{\dot{\alpha};\dot{\beta}} = Z^*_{\alpha;\beta}\,, \quad Z_{\alpha;\dot{\beta}} = Z^*_{\dot{\alpha};\beta}\,. \tag{2}$$

The basis now consists of the two independent dual spaces

$$Z_{\alpha;\beta}\,, \quad Z_{\alpha;\dot{\beta}}\,. \tag{3}$$

The question may be asked: are there any other conservation laws (apart from those connected with $\boldsymbol{c}$ and $\boldsymbol{c}'$) which have their geometrical origin in (3)? This question is equivalent with the question: are there any other groups which can be defined on the basis of (3) and which commute with $\boldsymbol{c}$ and $\boldsymbol{c}'$?

It can easily be shown that there are only two such groups: (i) the one-parametric commutative group $\boldsymbol{a}$

$$\boldsymbol{a}:\ Z'_{\alpha;\beta} = e^{i\varphi} Z_{\alpha;\beta}\,, \qquad Z'_{\alpha;\dot\beta} = e^{i\varphi} Z_{\alpha;\dot\beta}\,, \tag{4}$$

and (ii) another one-parametric commutative group $\boldsymbol{a}'$

$$\boldsymbol{a}':\ Z'_{\alpha;\beta} = e^{i\chi} Z_{\alpha;\beta}\,, \qquad Z'_{\alpha;\dot\beta} = e^{-i\chi} Z_{\alpha;\dot\beta}\,. \tag{5}$$

Thus, the simplest true representation (3) of $\boldsymbol{c}\cdot\boldsymbol{c}'$ provides at once a basis for $\boldsymbol{a}$ and $\boldsymbol{a}'$. The irreducible representations of the direct product

$$\boldsymbol{g} = \boldsymbol{c}\cdot\boldsymbol{c}'\cdot\boldsymbol{a}\cdot\boldsymbol{a}' \tag{6}$$

may be denoted by

$$F^{(n;m)}_{\alpha\ldots\beta\dot\gamma\ldots\dot\delta;\varrho\ldots\sigma\dot\lambda\ldots\dot\tau}\,, \tag{7}$$

where the lower indices before the semicolon transform with respect to $\boldsymbol{g}$ according to $\boldsymbol{c}$, and the indices after the semicolon according to $\boldsymbol{c}'$. The quantity (7) is symmetric in the indices before the semicolon as well as in those after the semicolon. The upper indices denote the transformation character with respect to $\boldsymbol{a}$ and $\boldsymbol{a}'$. The quantity $F^{(n;m)}$ transforms namely with respect to $\boldsymbol{g}$ according to

$$F^{(n;m)} = e^{i(n\varphi + m\chi)} F^{(n;m)}\,. \tag{8}$$

From the point of view of the whole group $\boldsymbol{g}$ we should write the basis (3) as

$$Z^{(1;1)}_{\alpha;\beta}\,, \qquad Z^{(1;-1)}_{\alpha;\dot\beta}\,. \tag{9}$$

However, the upper indices may be left out here if we remember only that to an undotted (dotted) lower index before the semicolon there corresponds the upper index 1 (-1) before the semicolon and, similarly, after the semicolon. This rule holds, of course, only for the basis (9) or (3) but not for an arbitrary representation (7).

It is now easy to establish such a correspondence between the fundamental particles and the representations (7) as to obtain a geometrical interpretation of all conservation laws within the framework of basis (3) (see Table I).

1. Baryon or nucleonic charge gauge transformation, connected with the baryon conservation law (N = number of baryons minus number of antibaryons), is represented by group $\boldsymbol{a}$ (cf. (4)).

TABLE I

Name	Representation	N (n)	Y (m)	$T_3 = M_3 - M_4$	Q	S	I_1	I_2	I_3	I_4	M_1	M_2	M_3	M_4
$\pi^{0\prime}$	$F^{(0;0)}$	0	0	0	0	0	0	0	0	0	0	0	0	0
K^+	$F^{(0;1)}_{;1}$	0	1	$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0
K^0	$F^{(0;1)}_{;2}$	0	1	$-\frac{1}{2}$	0	1	0	0	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	0
K^-	$F^{(0;-1)}_{;\dot{1}}$	0	-1	$-\frac{1}{2}$	-1	-1	0	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
$\tilde{K}^0$	$F^{(0;-1)}_{;\dot{2}}$	0	-1	$\frac{1}{2}$	0	-1	0	0	0	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$
π^-	$F^{(0;0)}_{;2\dot{1}}$	0	0	-1	-1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
π^+	$F^{(0;0)}_{;1\dot{2}}$	0	0	1	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
$\pi_3 - \pi_0'$	$F^{(0;0)}_{;1\dot{1}}$	0	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\pi_3 - \pi_0'$	$F^{(0;0)}_{;2\dot{2}}$	0	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
Λ^0	$F^{(1;0)}_{\alpha;}$ $F^{(1;0)}_{\dot{\alpha};}$	1	0	0	0	-1	$\frac{1}{2}$ 0	0 $\frac{1}{2}$	0 0	0 0	$\pm\frac{1}{2}$ 0	0 $\pm\frac{1}{2}$	0 0	0 0
P	$F^{(1;1)}_{\alpha;1}$ $F^{(1;1)}_{\dot{\alpha};1}$	1	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$ 0	0 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	0 0	$\pm\frac{1}{2}$ 0	0 $\pm\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	0 0
n	$F^{(1;1)}_{\alpha;2}$ $F^{(1;1)}_{\dot{\alpha};2}$	1	1	$-\frac{1}{2}$	0	0	$\frac{1}{2}$ 0	0 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	0 0	$\pm\frac{1}{2}$ 0	0 $\pm\frac{1}{2}$	$-\frac{1}{2}$ $-\frac{1}{2}$	0 0
Ξ^-	$F^{(1;-1)}_{\alpha;\dot{1}}$ $F^{(1;-1)}_{\dot{\alpha};\dot{1}}$	1	-1	$-\frac{1}{2}$	-1	-2	$\frac{1}{2}$ 0	0 $\frac{1}{2}$	0 0	$\frac{1}{2}$ $\frac{1}{2}$	$\pm\frac{1}{2}$ 0	0 $\pm\frac{1}{2}$	0 0	$\frac{1}{2}$ $\frac{1}{2}$
Ξ^0	$F^{(1;-1)}_{\alpha;\dot{2}}$ $F^{(1;-1)}_{\dot{\alpha};\dot{2}}$	1	-1	$\frac{1}{2}$	0	-2	$\frac{1}{2}$ 0	0 $\frac{1}{2}$	0 0	$\frac{1}{2}$ $\frac{1}{2}$	$\pm\frac{1}{2}$ 0	0 $\pm\frac{1}{2}$	0 0	$-\frac{1}{2}$ $-\frac{1}{2}$
Σ^-	$F^{(1;0)}_{\alpha;2\dot{1}}$ $F^{(1;0)}_{\dot{\alpha};2\dot{1}}$	1	0	-1	-1	-1	$\frac{1}{2}$ 0	0 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\pm\frac{1}{2}$ 0	0 $\pm\frac{1}{2}$	$-\frac{1}{2}$ $-\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$
Σ^+	$F^{(1;0)}_{\alpha;1\dot{2}}$ $F^{(1;0)}_{\dot{\alpha};1\dot{2}}$	1	0	1	1	-1	$\frac{1}{2}$ 0	0 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\pm\frac{1}{2}$ 0	0 $\pm\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$-\frac{1}{2}$ $-\frac{1}{2}$
$\Sigma_3 - \Lambda_0$	$F^{(1;0)}_{\alpha;1\dot{1}}$ $F^{(1;0)}_{\dot{\alpha};1\dot{1}}$	1	0	0	0	-1	$\frac{1}{2}$ 0	0 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\pm\frac{1}{2}$ 0	0 $\pm\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$
$-\Sigma_3 + \Lambda_0$	$F^{(1;0)}_{\alpha;2\dot{2}}$ $F^{(1;0)}_{\dot{\alpha};2\dot{2}}$	1	0	0	0	-1	$\frac{1}{2}$ 0	0 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\pm\frac{1}{2}$ 0	0 $\pm\frac{1}{2}$	$-\frac{1}{2}$ $-\frac{1}{2}$	$-\frac{1}{2}$ $-\frac{1}{2}$

2. Hypercharge gauge transformation, connected with the isospinor conservation law ($Y =$ number of isospinors minus number of antiisospinors), is represented by group $\boldsymbol{a}'$ (cf. (5)).

3. Rotation T_3 around the third axis in the isotopic spin space corresponds to the following transformation of $\boldsymbol{c}'$ for $Z_{\alpha;\beta}$

$$\begin{pmatrix} Z'_{\alpha;1} \\ Z'_{\alpha;2} \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}\varphi} & 0 \\ 0 & e^{-\frac{i}{2}\varphi} \end{pmatrix} \begin{pmatrix} Z_{\alpha;1} \\ Z_{\alpha;2} \end{pmatrix} \tag{10}$$

and the same transformation for $Z_{\alpha;\dot{\beta}}$.

4. Conservation of charge Q (in units of e) corresponds to the product of (5) with (10), with $\chi = \varphi/2$

(11) $$Q \doteq T_3 + \frac{Y}{2}.$$

5. Conservation of strangeness S corresponds to the product of (5) with (4), with $\varphi = -\chi$

(12) $$S = Y - N.$$

6. Conservation of angular momentum is connected with transformations of $\boldsymbol{c}'$.

7. Conservation of energy-momentum (connected with translations in Minkowski's space) may be obtained when the connexion between the co-ordinate differences in the Minkowski space and the variables $z_{\alpha;\beta}$, $z_{\alpha;\dot\beta}$ of the spinor space is established (cf. [1]-[3]).

Having thus obtained a geometrical interpretation of conservation laws in the framework of the spinor space (3) questions may be put relative to the choice of representations occurring in Table I. Without pretending to explain, why only some of the possible representations are realized in nature, it is possible to establish some simple restrictive rules. For this purpose it is convenient to write explicitly the basic vectors of the general representation (7)

(13) $$w^{I_1,I_2,I_3,I_4,n,m}_{M_1,M_2,M_3,M_4} = \frac{u_1^{I_1+M_1}u_2^{I_1-M_1}u_{\dot 1}^{I_2+M_2}u_{\dot 2}^{I_2-M_2}v_1^{I_3+M_3}v_2^{I_3-M_3}v_{\dot 1}^{I_4+M_4}v_{\dot 2}^{I_4-M_4}e_1^n e_2^m}{\sqrt{(I_1+M_1)!(I_1-M_1)!(I_2+M_2)!(I_2-M_2)!(I_3+M_3)!(I_3-M_3)!(I_4+M_4)!(I_4-M_4)!}}$$

where u_α, $u_{\dot\alpha}$ transform with respect to $\boldsymbol{g}$ according to $\boldsymbol{c}$, v_α $v_{\dot\alpha}$ according to $\boldsymbol{c}'$, e_1 according to $\boldsymbol{a}$ and e_2 according to $\boldsymbol{a}'$.

It is now easily seen that we obtain the transformation types listed in Table I if we restrict the values of I_i to

(14) $$I_1 + I_2 \leqslant \tfrac{1}{2}, \quad I_3 \leqslant \tfrac{1}{2}, \quad I_4 \leqslant \tfrac{1}{2}$$

and the values of n and m to

(15) $$n \equiv N = \begin{cases} 2(|M_1| + |M_2|) & \text{for particles} \\ -2(|M_1| + |M_2|) & \text{for antiparticles} \end{cases}$$

(16) $$m \doteq Y = 2(|M_3| - |M_4|) \quad \text{for both particles and antiparticles}$$

It may be noted that in the framework of the full group $\boldsymbol{c}'$ we obtain two possibilities for Λ: (i) a $\boldsymbol{c}'$ — scalar and (ii) the fourth component of a $\boldsymbol{c}'$ — vector. In the framework of the unitary subgroup $\boldsymbol{u}'$ of $\boldsymbol{c}'$ these two possibilities coincide. A similar situation presents itself with respect to the neutral π-meson.

When only the unitary subgroup $\boldsymbol{u}'$ of $\boldsymbol{c}'$ is considered another possibility offers itself to define $\boldsymbol{g}$ as the product $\boldsymbol{c}\cdot\boldsymbol{u}'\cdot\boldsymbol{u}''\cdot\boldsymbol{a}=\boldsymbol{c}\cdot\boldsymbol{E}_4\cdot\boldsymbol{a}$ ($\boldsymbol{E}_4=$ the group of four-dimensional Euclidean rotations). Indeed, in this case the variables $Z^{(1;1)}_{\alpha;1}, Z^{(1;1)}_{\alpha;2}$ transform like $-Z^{(1;1)}_{\alpha;2}, Z^{(1;1)}_{\alpha;1}$, respectively and the variables $Z^{(1;-1)}_{\alpha;1}, Z^{(1;-1)}_{\alpha;2}$ transform like $Z^{(1;-1)}_{\alpha;2}, -Z^{(1;-1)}_{\alpha;1}$, respectively: We may, therefore, define another unitary group $\boldsymbol{u}''$ (commuting with $\boldsymbol{c}, \boldsymbol{u}'$ and $\boldsymbol{a}$)

$$(17)\qquad \boldsymbol{u}'':\begin{pmatrix} Z^{(1;1)'}_{\alpha;\beta} \\ Z^{(1;-1)'}_{\alpha;\beta}\end{pmatrix}=\begin{pmatrix} a & b \\ -b^* & a^*\end{pmatrix}\begin{pmatrix} Z^{(1;1)}_{\alpha;\beta} \\ Z^{(1;-1)}_{\alpha;\beta}\end{pmatrix},$$

which contains $\boldsymbol{a}'$ as a subgroup. To carry out the geometrization we need only $\boldsymbol{c}\cdot\boldsymbol{u}'\cdot\boldsymbol{a}\cdot\boldsymbol{a}'$ so that it is unimportant (from this point of view), whether we generalize this group to $\boldsymbol{c}\cdot\boldsymbol{c}'\cdot\boldsymbol{a}\cdot\boldsymbol{a}'$ or $\boldsymbol{c}\cdot\boldsymbol{u}'\cdot\boldsymbol{u}''\cdot\boldsymbol{a}=\boldsymbol{c}\cdot\boldsymbol{E}_4\cdot\boldsymbol{a}$. It may be noted that $\boldsymbol{c}\cdot\boldsymbol{c}'\cdot\boldsymbol{a}\cdot\boldsymbol{a}'$ does not mix the two dual spaces (3) in contradistinction to $\boldsymbol{c}\cdot\boldsymbol{E}_4\cdot\boldsymbol{a}$.

By making the condition for I_i less restrictive:

$$(18)\qquad I_i \leqslant \tfrac{1}{2} \quad (i=1,\dots,4)$$

we get three new transformation types, namely a $\boldsymbol{c}$-vector – $\boldsymbol{c}'$-scalar, $\boldsymbol{c}$-vector – $\boldsymbol{c}'$-spinor and $\boldsymbol{c}$-vector – $\boldsymbol{c}'$-vector. The first one may be considered to describe the photon. The other two are not directly observed (one could imagine them to transmit weak interactions).

The leptons may be obtained with

$$(19)\qquad I_1+I_2\leqslant \tfrac{1}{2}, \quad I_3+I_4\leqslant\tfrac{1}{2}$$

and a properly chosen connection between n, m and M_1, M_2; M_3, M_4. Since a unique isotopic characterization of leptons does not exist (there are several different attempts), we will not discuss here these possibilities in detail.

It may be mentioned that restriction (18) is equivalent with replacing the vectors $u_\alpha, u_{\dot\alpha}, v_\alpha, v_{\dot\alpha}$ by operators $u_\alpha\xi_1, u_{\dot\alpha}\xi_2, v_\alpha\xi_3, v_{\dot\alpha}\xi_4$, with ξ_i satisfying relations

$$(20)\qquad \xi_i^2=0, \ [\xi_i,\xi_k]_-=0, \quad (i,k=1,\dots 4),$$

or

$$(21)\qquad \xi_i^2=0, \ [\xi_i,\xi_k]_+=0, \quad (i,k=1,\dots 4),$$

respectively. Restriction (14) is obtained herefrom by putting $\xi_1=\xi_2$, restriction (19) by putting $\xi_1=\xi_2$ and $\xi_3=\xi_4$.

It is obvious from the above considerations that the choice of the geometrical basis (3) is unique, if we restrict ourselves to the lowest true representation of the group $\boldsymbol{g}$. There are, of course, other possibilities among the higher representations of $\boldsymbol{g}$. These possibilities may be excluded since they correspond to spaces with unnecessarily many dimensions. There

are also possibilities connected with untrue representations (as e.g. $F_{\alpha\dot{\beta};}$ and $F_{;\alpha\dot{\beta}}$). They exhibit the common disadvantage of translational degrees of freedom (plane waves) in the iso-space. These degrees of freedom must be excluded by certain additional conditions which possess no justification in the geometry. With the basis (3) such difficulties do not occur, as shown explicitly in [3].

Therefore, it is tempting to consider the fields describing the fundamental particles as functions of the variables $Z_{\alpha;\beta}$, $Z_{\alpha;\dot{\beta}}$ of the spinor space and to formulate dynamics in this space. Some of the consequences of such an attempt were described in [1]-[3]. In particular one obtains differential equations ([1]-[3]) which possess a larger manifold of solution than the conventional Dirac or Klein-Gordon equations and which may be connected with the structure of elementary particles. One may also show that in the framework of the spinor space (3) inversions P never occur separately but always in connection with charge conjugation ([2], and [3]). This provides a possibility for a natural explanation of parity non-conservation. A detailed discussion of the latter problem will be the subject of a subsequent paper.

Abstract

A geometrical interpretation of the conservation laws of angular momentum, isotopic spin, baryon number, charge and hypercharge is given in the framework of the spinor space. Some rules restricting the possible representations to those which correspond to observed particles are discussed. It is shown that, with certain natural assumptions, the spinor space is the only one which may be used for geometrical interpretation of the whole set of known conservation laws.

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THEORETICAL PHYSICS

On the Interpretation of Isovector Components

by

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Description of $\frac{1}{2}$-spin fields with the help of 4-spinors gives the possibility to introduce the transformation group Λ'. This group commutes with the representations Λ of the 6-parameter proper Lorentz group. Λ' induces rotations in some space called conventionally the isospace.

Transformations Λ' were introduced first by Pauli [1] for neutrino equation and by Gürsey [2] for the nucleon field. These transformations were also used by Rzewuski [3], [4] in his spinor description of space-time co-ordinates.

Generally Λ' (with the additional condition of unimodularity) generates a four-dimensional Minkowski isospace [2], [3]. This isospace is admissible only by neutrino equation *). If $\varkappa \neq 0, e = 0$, we obtain the result that the corresponding equation admits transformations in the $(1, 2, 4)$-subspace of the full isospace. In the general case, if $e, A_\mu \neq 0$, the Dirac equation permits only rotations in $(1, 2)$-isoplane. By these considerations we show that isotransformations can be interpreted as an arbitrariness in definition of charge and mass conjugation, which are allowed by the equation.

Isocomponents are useful in investigation of the full Lorentz group. It is shown that the isocomponents of tensors correspond to different transformation types with respect to inversions of space and time.

1. Dirac's equation with the electromagnetic field A_μ:

$$\{\gamma_\mu(\partial_\mu - \iota e A_\mu) + \varkappa\}\psi = 0 \tag{1}$$

is invariant with respect to the full Lorentz group. The proper Lorentz group is realised on ψ by means of the following transformations:

$$\psi' = \Lambda\psi = (\exp[\tfrac{1}{2}\sigma_{\mu\nu}a_{\mu\nu}])\psi\,,$$

where $a_{\mu\nu}a_{\mu\varrho} = \delta_{\nu\varrho}$.

*) Pauli's restriction to $(1, 2, 3)$-subspace is connected with invariance of the commutation relations. Here we consider only classical spinor fields.

The group of inversions $I=(1, I_r, I_t, I_{rt})$ is obtained with help of the well-known matrices *):

$$\begin{aligned} I_r: &\quad \psi' = a_r\gamma_4\psi \\ I_t: &\quad \psi' = a_t\gamma_5\gamma_4\psi \end{aligned} \qquad a_r^4 = a_t^4 = 1 \,.$$

Thus, we have four types of spinors:

$$\text{(2)} \qquad \begin{cases} a: & a_r = \pm 1 \quad a_t = \pm 1 \\ b: & a_r = \pm \iota \quad a_t = \pm \iota \\ c: & a_r = \pm 1 \quad a_t = \pm \iota \\ d: & a_r = \pm \iota \quad a_t = \pm 1 \end{cases}$$

It is easily seen, that there are only two independent operators $\hat{C}$ and $\hat{M}$, which commute with Λ:

$$\hat{C}\psi = C\overline{\psi}\,, \qquad C\gamma_\mu^T = \gamma_\mu C$$
$$\hat{M}\psi = M\psi\,, \qquad M\gamma_\mu = -\gamma_\mu M$$

$\hat{C}$ and $\hat{M}$ are charge and mass conjugation operators with respect to Eq. (1). This implies that:

$$\hat{C}^2 = \hat{M}^2 = 1\,,$$

and we have

$$C^+C = 1 \qquad M = \gamma_5\,.$$

The most general operator Λ', which commutes with Λ, is:

$$\text{(3)} \qquad \Lambda' = a + b\hat{M} + c\hat{C} + d\hat{M}\hat{C}\,.$$

2. We construct an arbitrary hermitean scalar S with respect to Λ. We obtain:

$$S = \overline{\psi}\Lambda'\psi + \text{H. C.}$$

If we write the following six hermitean scalar forms $Y_{;i\varrho}$:

$$\begin{aligned} Y_{;0} &= \overline{\psi}\psi\,, \\ Y_{;3} &= \iota\overline{\psi}\gamma_5\psi\,, \\ Y_{;1} + iY_{;2} &= \overline{\psi}\hat{C}\psi\,, \\ Y_{;4} + \iota Y_{;5} &= \overline{\psi}\gamma_5\hat{C}\psi\,, \end{aligned}$$

we have:

$$S = s_{;\varrho}Y_{;\varrho} \qquad s_{;\varrho} - \text{real}; \qquad \varrho = 0, 1, \ldots, 5\,.$$

*) It is assumed here that x_μ and A_μ have the same character with respect to inversions ($\partial_\mu A_\mu$ is a pure scalar).

It is easy to construct analogous six types of hermitean vectors $Y_{\mu;\varrho}$:

(4)
$$\begin{cases} Y_{\mu;0} = \iota\bar{\psi}\gamma_\mu\psi\,, \\ Y_{\mu;3} = \iota\bar{\psi}\gamma_5\gamma_\mu\psi\,, \\ Y_{\mu;1} + \iota Y_{\mu;2} = \bar{\psi}\gamma_\mu\hat{C}\psi\,, \\ Y_{\mu;4} + \iota Y_{\mu;5} = \bar{\psi}\gamma_5\gamma_\mu\hat{C}\psi\,. \end{cases}$$

The most general hermitean vector with respect to Λ constructed from the 4-spinor ψ has the form:

(5)
$$W_{\mu;} = \omega_{;\varrho} Y_{\mu;\varrho} \qquad \omega_{;\varrho} - \text{real}; \qquad \varrho = 0, 1, \dots, 5\,.$$

The bilinear forms, which are tensors of second rank can be written similarly:

$$T_{\mu\nu;} = t_{;\varrho} Y_{\mu\nu;\varrho}\,, \qquad t_{;\varrho} - \text{real}; \qquad \varrho = 0, 1, \dots, 5\,,$$

where the expressions for $Y_{\mu\nu;\varrho}$ are (except for a factor i) just expressions (4) with γ_μ replaced by $\gamma_\mu\gamma_\nu$.

Let us express $Y_{;\varrho}$, $Y_{\mu;\varrho}$ and $Y_{\mu\nu;\varrho}$ by means of Weyl's two-component spinors c_a and $c_{\dot a}$. Using:

$$\gamma_\iota = \varrho_2 \Sigma_\iota\,, \qquad \gamma_4 = \varrho_1\,, \qquad \gamma_5 = \varrho_3\,,$$
$$\hat{C} c_a = \lambda c^*_{\dot a}\,, \qquad \hat{C} c_{\dot a} = \lambda c^*_a\,, \qquad |\lambda|^2 = 1\,,$$

we obtain in classical theory $Y_{;1} = Y_{;2} = Y_{;4} = Y_{;5} = 0$ and $Y_{\mu;4} = Y_{\mu;5} \equiv 0$.

Now we shall investigate only $Y_{\mu;\varrho}$. The first four non-vanishing $Y_{\mu;\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ can be expressed simply with the help of a new 4-spinor $\hat{\psi}$. Choosing $\lambda = \iota$ we have:

(6)
$$Y_{\mu;\nu} = \tilde{\psi} \Sigma_\mu \varrho_\nu \tilde{\psi}\,,$$

where $\tilde{\psi} = \begin{pmatrix} c_a \\ c^*_{\dot a} \end{pmatrix}$.

By virtue of (6), (5) reduces to:

(7)
$$W_{\mu;} = \omega_{;\nu} Y_{\mu;\nu}\,.$$

3. Because of

$$[\Lambda, \Lambda'] = 0$$

we can call Λ' the group of isotransformations. Λ' can be expressed on ψ

(8)
$$\psi' = (a + b\gamma_5)\psi + (c + d\gamma_5)\hat{C}\psi$$

or on $\tilde{\psi}$:

$$\tilde{\psi}' = (a_\iota \varrho_\iota + a_0)\tilde{\psi}\,, \qquad \iota = 1, 2, 3$$
$$a_0 = a_1 + \iota b_2\,,$$
$$a_3 = b_1 + \iota a_2\,,$$
$$a_2 = c_2 + \iota d_1\,,$$
$$a_1 = -c_1 - \iota d_2\,,$$

where $a = a_1 + ia_2$ etc.

For unimodular Λ'

(9) $$|\Lambda'| = a_0^2 - a_\iota^2 = 1$$

the index after the semicolon in (6) denotes isovector components in Minkowski's $E_4^{(1)}$-isospace. Thus, $Y_{\mu;\nu}$ indicates the vector-isovector component of a mixed tensor in two spaces. The unimodular Λ' describes rotations in isospace and Λ — usual Lorentz rotations.

4. The general 6-parameter transformation Λ' is limited by the form of the equation. Because the $\hat{C}$ and $\hat{M}$ from (3) have a physical interpretation, we can treat the isotransformations, which are allowed by Eq. (1), as an arbitrariness in definition of charge and mass conjugation operators. Eq. (1) admits only:

(10) $$\begin{cases} \hat{M}' = \hat{M}\,, \\ \hat{C}' = e^{i\alpha}\hat{C}\,. \end{cases}$$

From (4) we obtain that (10) induces:

(11) $$\begin{cases} Y'_{\mu;1} = \cos\alpha\, Y_{\mu;1} - \sin\alpha\, Y_{\mu;2}\,, \\ Y'_{\mu;2} = \sin\alpha\, Y_{\mu;1} + \cos\alpha\, Y_{\mu;2}\,, \\ Y'_{\mu;3} = Y_{\mu;3}\,, \\ Y'_{\mu;0} = Y_{\mu;0}\,. \end{cases}$$

We see, that only isorotations in (1, 2)-plane leave (1) unchanged. This one-parameter group is generated by the I_3-component of the isospin vector $\vec{I}$, which generates rotations in the Euclidean subspace (1, 2, 3) of $E_4^{(1)}$.

We can obtain rotation (11) also from (6) with the help of the following transformation on $\tilde{\psi}$:

(12) $$\tilde{\psi}' = e^{i\frac{\alpha}{2}\varrho_3}\tilde{\psi}\,.$$

It may be interesting to mention that if in (1) $e = 0$ (or $A_\mu = 0$), we obtain as the only possible transformations in $E_4^{(1)}$ the group

(13) $$\tilde{\psi}' = u_{(3)}\tilde{\psi}\,,$$

where

$$u_{(3)} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}.$$

If (9) is satisfied, we obtain the result that the free Dirac equation admits a three-dimensional pseudoeuclidean isospace $E_3^{(1)}$, which is a (1, 2, 4)-subspace of the full Minkowski's isospace. Eq. (13) may be also expressed by means of ψ:

$$\psi' = a\psi + b\hat{C}\psi\,,$$

$$|a|^2 - |b|^2 = 1\,.$$

Since the third isoaxis is invariant under $U_{(3)}$, we have from (4) $\hat{M}' = \hat{M}$. All transformations in $E_3^{(1)}$ are caused by a change of the charge conjugation operator.

For $\varkappa = 0, e \neq 0$ we obtain, by virtue of (9):

$$\widetilde{\psi}' = e^{(\beta+i\alpha)\varrho_3}\widetilde{\psi} \qquad \alpha, \beta - \text{real}\,.$$

The parameter β induces pseudoeuclidean rotations in the $(3, 4)$-isoplane. They are caused by changes of the mass conjugation operator.

Eq. (1) with $\varkappa = 0, e = 0$ (neutrino) is invariant with respect to the full isotopic group.

If we consider ψ as a field operator, only the $(1, 2, 3)$-subspace of $E_4^{(1)}$ leaves the conventional commutation relations invariant. Thus, we obtain for neutrino a three-parameter Pauli group, and for Dirac particles with $\varkappa \neq 0$ or $e \neq 0$ only a one-parameter phase factor group (12).

4. Now we shall investigate the properties of $Y_{\mu;\nu}$ with respect to the group of inversions. It is easily seen that these properties are determined by the index behind the semicolon (see the following Table).

	Type of spinor (see (2))	Kind of vector (see (*))
$Y_{\mu;0}$	a, b, c, d	C
$Y_{\mu;3}$	a, b, c, d	D
$Y_{\mu;1}$ or $Y_{\mu;2}$	a	B
	b	A
	c	D
	d	C

(*) A, B, C, D — well-known tensor types with respect to inversions [5].

Let us consider an arbitrary form (7). In this form the $\omega_{\mu;}$ are components of an isovector. Properties of $W_{\mu;}$ with respect to inversions are determined by this isovector. Thus, if we want to describe the behaviour of an arbitrary vector with respect to the full Lorentz group, we must carry out the decomposition (7).

We see from the Table, that rotations in the $(1, 2)$-isoplane are distinguished also with respect to the inversions group. They are namely the only isotransformations, which do not change the inversion types.

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EXPERIMENTAL PHYSICS

The Polarization of Neutrons from the $^{12}C(d, n)^{13}N$ Reaction

by

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Introduction

The polarization of nucleons from the stripping reaction of deuterons has been theoretically predicted and calculated by Newns [14], Horowitz and Messiah [8], Cheston [3], Sawicki [17], Newns and Refai [15], Sachtler [18], and others. Experimentally this polarization effect was observed the first time by Hillman [6] in 1956 for ground state protons in the reaction $^{12}C(d, p)^{13}C$. Later this polarization has been detected in several cases of the reaction (d, p), when using the following target nuclei: ^{9}Be, ^{10}B, ^{12}C, ^{28}Si and ^{40}Ca [2], [5], [7], [9], [10]. As regards experimental evidence of the polarization of neutrons from the (d, n) reaction, there only exists a single short abstract by Haeberli and Rolland in the programme of the Washington Meeting (April 1957) of the American Physical Society [4]. These authors investigated the polarization of neutrons in the $^{12}C(d, n)^{13}N$ reaction at deuteron energies from 2.5 to 3.6 MeV. This means that in these experiments neutrons from the (d, n) reaction corresponded to the ground energy state of the ^{13}N nucleus.

The polarization in stripping reaction is theoretically explained by the distortion of the plane wave of the incoming deuteron and that of the outgoing nucleon. The sign of the polarization caused by the distortion of the plane wave of the deuteron is opposite to that of polarization caused by the distortion of the plane wave of the outgoing nucleon [15], [18]. The final polarization effect is practically obtained by simple addition of both these effects. Experimental results obtained so far indicate in most cases, for $l = 1$ of the captured particle, the preponderance of the deuteron plane wave distortion.

The degree of polarization of the outgoing nucleon in the stripping reaction supplies information concerning the mechanism of the reaction under consideration, and the extent of the contribution of the compound

nucleus formation in this reaction. It seems probable that, if an exact knowledge of the mechanism of the polarization in stripping reactions is obtained, the sign of the polarization will be of great help in distinguishing between $l+\frac{1}{2}$ and $l-\frac{1}{2}$ values of the spin of the energy levels of the resulting nucleus.

The present report relates to the preliminary results of the investigation of polarization of neutrons in the $^{12}C(d, n)^{13}N$ reaction. These results are limited to the measurements of the polarization of neutrons at a single reaction angle $\theta_{lab} = 15°$ and the energy of deuterons $E_d \doteq 12.9$ MeV.

Experimental arrangement

The beam of 12.9 MeV deuterons from the 120 cm Cracow cyclotron passing quadrupole magnetic lenses and a deflecting magnet was focussed on a target situated in the target hall about 11 m from the cyclotron magnet (see Fig. 1). The intensity of

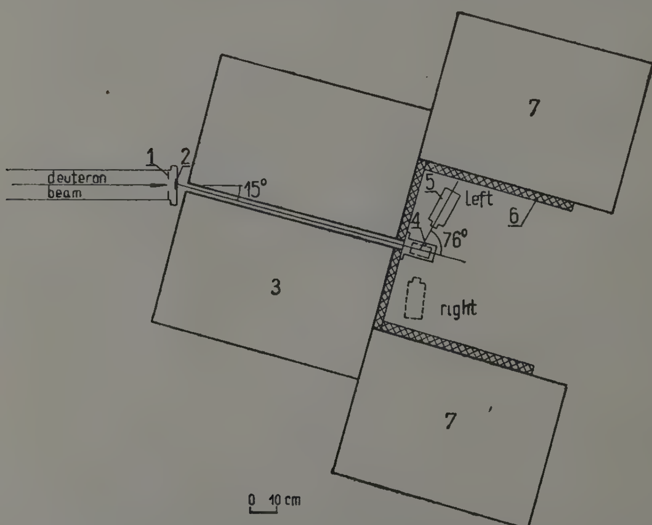

Fig. 1. Experimental arrangement: *1*. Tungsten ring; *2*. Carbon target; *3*. Paraffin wax collimator; *4*. Proportional counter filled with helium; *5*. Hornyak scintillation counter; *6*. Lead shielding; *7*. Water shielding

the ion current in the course of the reported experiments was $3-4\,\mu A$. The target consisted of a water cooled 0.3 mm thick tungsten foil covered with a layer of carbon about 7 mg/cm² thick. The angle of the reaction $\theta_{lab} = 15°$ corresponding to $\theta_{c.m.} = 17°$ has been chosen in connection with the theoretically anticipated comparatively large intensity of the neutron beam and different from zero polarization at this angle [13], [18].

The process of $n-\alpha$ scattering was used as the polarization analyser. Due to a high level of the fast neutrons background and the necessity to distinguish between neutrons

belonging to different energy groups, the coincidence method was applied. Neutrons elastically scattered on nuclei of helium filling a proportional counter were registered using a Hornyak type fast neutron scintillation counter. Coincidences between helium recoil nuclei and scattered neutrons were registered for both the right and left position of the Hornyak counter. The proportional counter was filled with spectroscopically pure helium at the pressure of 11 at. The scattering angle of neutrons was 90^0 in c. m. co-ordinate system. The measurements were controlled by a fixed number of counts of the proportional counter. The background produced by accidental coincidences was determined by periodical switching on and off of a 25 microsecond delay circuit between the Hornyak counter and the coincidence arrangement.

Due to the poor quality of our polarization analyser a more exact separation of neutron groups of different energies was not possible. However, the analysis of the spectrum of coincidence impulses from the proportional counter has shown a very strong relative abundance of neutrons corresponding to the 3.58 MeV excited level of the ^{13}N nucleus. The results obtained by Middleton et al. [13] for $E_d = 8.1$ MeV, corrected to correspond to the efficiency of our proportional counter, give for relative numbers of neutrons corresponding to the second, first and ground levels in the ^{13}N nucleus the following figures: 1:0.3:0.14. According to our results for $E_d = 12.9$ MeV the neutrons corresponding to the second excited level of the ^{13}N nucleus are relatively still more numerous. This is compatible with the results of measurements of energy spectrum of protons from the reaction $^{12}C(d,p)^{13}C$ for $E_d = 14$ MeV [12].

Auxiliary measurements performed using a bare tungsten target without any carbon on it have shown that the background effect of neutrons from the stripping reaction of deuterons on tungsten does not exceed 10% of the effect obtained for the target covered with a layer of carbon.

Results

The degree of polarization of neutrons evaluated from the left and right asymmetry of scattering on helium nuclei in the plane of the reaction is $-(0.39 \pm 0.11)$. As the positive direction of polarization the direction of the product $\boldsymbol{K}_{\mathrm{in}} \times \boldsymbol{K}_{\mathrm{out}}$ is adopted, where $\boldsymbol{K}_{\mathrm{in}}$ and $\boldsymbol{K}_{\mathrm{out}}$ indicate the wave-vector of the incident and outgoing particle respectively.

Measurements performed in order to check the apparatus and the experimental method applied gave the following results for the up and down asymmetry determined as:

$$\frac{N_{\mathrm{above}} - N_{\mathrm{below}}}{N_{\mathrm{above}} + N_{\mathrm{below}}} = 0.03 \pm 0.04,$$

where N_{above} and N_{below} indicate the number of coincidences obtained with the scintillation counter above and below the reaction plane perpendicularly to this plane respectively.

The degree of polarization of neutrons elastically scattered in helium was determined from the phase shift analysis perfomed by Seagrave [19] and Levintov [11].

Discussion

The energy level 3.56 MeV in ^{13}N nucleus which is mostly responsible for the neutrons being observed is a single particle level with $l = 2$.

Taking into account the results obtained by Juveland and Jentschke [10], Hensel and Parkinson [5], Hillman [6], and Hird [7] for the level of the resulting nucleus with $l = 1$ the negative sign of polarization suggests for the energy level 3.56 MeV of ^{13}N the value of spin $3/2^+$.

Alternatively, the negative sign of the polarization of protons from the reaction ^{12}C$(d, p)^{13}$C corresponding to the second excited level $5/2^+$ of the ^{13}C nucleus with $l = 2$ permits us to suppose that in both these cases the negative sign of the polarization determines the value $j = l + 1/2$. This could be connected with the preponderance of the distortion of the distortion of the wave of the outgoing nucleon.

The polarization obtained in the present work considerably exceeds the value calculated by Newns and Refai [15]. This discrepancy is similar to that in the experiments of Hillman [6], who for the polarization of protons from the reaction ^{12}C$(d, p)^{13}$C at $E_d = 4.05$ MeV and $\theta_{\text{c.m.}} = 30°$ obtained the value $-(0.58)$. It is possible that this high value of polarization is connected with the small difference between the energy of the outgoing nucleons and that of the resonance scattering of these nucleons.

A more exact analysis of the effects under consideration will be possible after the angular and energy dependence of the polarization of neutrons from the reaction (d, n) is known. Further measurements with the aim of fulfilling this programme are now in progress.

The authors are indebted to the Cyclotron Operation Staff for their continuous work in maintaining and improving the cyclotron operation conditions, Dr. E. Zieliński from the School of Mining and Metallurgy, Cracow, for filling our proportional counters with spectroscopically pure helium, as well as Mr. A. Radwan and Mr. Z. Wroński for their substantial help in preparing the apparatus.

Abstract

The polarization of neutrons emitted from the stripping reaction ^{12}C$(d, n)^{13}$N has been investigated at the reaction angle $\theta_{\text{lab}} = 15°$ and deuteron energy $E_d = 12.9$ MeV. The polarization of neutrons connected with the 3.56 MeV energy level in ^{16}N nucleus was found to be $-(0.39 \pm 0.11)$.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

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К. ГЕМБА, А. ГРАНАС, А. ЯНКОВСКИЙ, ТЕОРЕМА О ВЫМЕТАНИИ В БАНАХОВЫХ ПРОСТРАНСТВАХ . . . . стр. 539—544

Известная теорема К. Борсука о выметании обобщается на случай произвольных банаховых пространств. Простым следствием доказываемой теоремы является известная теорема Биркгоффа-Келлога о существовании собственных значений для нелинейных операторов в банаховых пространствах.

Ч. РЫЛЛЬ-НАРДЗЕВСКИЙ, О КАТЕГОРИЧНОСТИ В МОЩНОСТИ $\leqslant \aleph_0$ . . . . стр. 545—548

Предметом работы является понятие категоричности в мощности, введенное И. Лосем [1] и Р. Л. Вогтом [4]. Рассматривается элементарная теория Т, не-противоречивая и полная, опирающаяся на узком исчислении предикатов с тождеством. Полагается, что в моделях теории Т интерпретация тождества всюду натуральна.

Целью работы — дать характеристику теорий Т таких, что они категоричны в мощности $\leqslant \aleph_0$ (т. е. таких, для которых каждые две модели мощности $\leqslant \aleph_0$ изоморфны). Для этого необходимым и достаточным условием является следующее:

Для каждого натурального n число разных отношений n-переменных, элементарно определимых в теории Т — конечно.

(Через „элементарно определимые отношения" мы понимаем отношения определимые при помощи узкого исчисления предикатов из первичных понятий теории Т).

Доказательство достаточности аналогично классической теореме о характеристике порядка η; доказательство необходимости более сложно.

Результаты, приведенные в работе, были предложены автором на заседании Польского Математического Общества, Отделение в Торуни, в 1955 г. ([2], стр. 24).

В. МАТУШЕВСКАЯ и В. ОРЛИЧ, О ЛОКАЛЬНЫХ КРИТЕРИЯХ СХОДИМОСТИ РЯДОВ ФУРЬЕ . . . . стр. 549—556

В различных вопросах анализа приходится рассматривать классы непрерывных функций, по отношению к которым полагается добавочно, что они

обладают некоторыми локальными свойствами, как напр. условие Гельдера, ограниченная вариация в некоторой окружности и т. д. Для каждого локального свойства W имеется соответствующая тройка $[x, \xi, \delta]$, где x — обозначает непрерывную функцию, принадлежащую к классу K и обладающую свойством W в окружности $(\xi-\delta, \xi+\delta)$.

В работе рассматриваются условия, при соблюдении которых справедлива следующая теорема (T): *в классе K имеются функции, которые ни в какой окружности некоторого счетного множества точек не обладают свойством* W.

Дается некоторая система постулатов на тройке $[x, \xi, \delta]$, которых теорема (T) справедлива для класса K, состоящего из рядов, имеющих вид $a_1\varphi_1(t)+a_2\varphi_2(t)+\ldots+a_n\varphi_n(t)+\ldots$, где $|a_1|+|a_2|+\ldots+|a_n|+\ldots<\infty$, причем $\varphi_n(t)$ — непрерывные функции, совместно ограниченные, удовлетворяющие некоторым добавочным условиям.

В результате своих рассуждений авторы получили теорему, согласно которой некоторый тип локальных критериев сходимости рядов Фурье не может дать конечных условий сходимости.

Я. КУРЦВЕЙЛЬ, ЛИНЕЙНЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ С ОБОБЩЕННЫМИ ФУНКЦИЯМИ В КАЧЕСТВЕ КОЭФФИЦИЕНТОВ . . . . стр. 557—560

В работе дается формулировка теоремы существования и единственности для системы:

$$\frac{dx_i}{dt}=\sum_{j=1}^{k} a_{ij}x_j\,, \qquad i=1,2,\ldots,k \tag{1}$$

при предположении, что в качестве некоторых коэффициентов a_{ij} допустимы обобщенные функции (не очень высоких порядков) если другие коэффициенты — достаточно гладкие функции. „Начальное условие" рассматривается в виде $\varphi_i x_i=\varphi_i$, где φ_i — линейный функционал на подходящем пространстве и x_i — действительное число. Точные результаты требуют подробных обозначений. Рассматривается вопрос о существовании значений решений x_i в точке t_0, а также вопрос о существовании и единственности решения x уравнения

$$x^{(k)}+a_1x^{(k-1)}+,\ldots,+a_kx=0\,,$$

удовлетворяющего условию $x(t_i)=\lambda_i$, $i\neq 1,2,\ldots,k$.

А. Т. БАРУХА-РЕЙД, ЗАМЕТКА ОБ УРАВНЕНИЯХ ПРОИЗВОЛЬНЫХ ОПЕРАТОРОВ В ПРОСТРАНСТВАХ БАНАХА . . . . . стр. 561—564

В работе приводятся результаты вступительных исследований уравнения произвольных операторов

$$\big(T(\omega)-\lambda I\big)x(u,\omega)=y(u,\omega)\,,$$

где x и y являются обобщенными произвольными переменными (переменные элементы) в разделимом Банаховом пространстве X, а также $T(\omega)$ — измеримым произвольным эндоморфизмом на X.

Полученные автором результаты относятся к (i) существованию и измеримости произвольного оператора решения $R_\lambda(T, \omega) = (T(\omega) - \lambda I)^{-1}$; (ii) рядовому представлению оператора $R_\lambda(T, \omega)$ для множества $T(\omega)$, являющегося решением уравнения; (iii) существованию измеримого произвольного решения $x(u, \omega) = R_\lambda(T, \omega) y(u, \omega)$ упомянутого операторного уравнения.

В работе дается геометрическая интерпретация законов сохранения момента количества движения изотопического спина, числа барионов, заряда в пространстве спинорных переменных.

Дискутируются правила, ограничивающие количество репрезентаций до типов, соответствующих наблюдаемым частицам.

Показано, что при некоторых естественных предположениях, спинорное пространство является единственным пространством, пригодным для геометризации законов сохранения.

В работе рассматривается уравнение Дирака с точки зрения изотопических преобразований. Автором была построена, в первую очередь, полная изогруппа Минковского и соответствующие изовекторы. Изо-преобразования, допустимые уравнением поля, вводятся как произвольные элементы в определение операторов сопряжения заряда и массы.

Показано, что разложение обыкновенного вектора на его изо-составляющие равносильно его разложению на преобразования разных типов относительно инверсий.

Исследовалась поляризация нейтронов, испускаемых в реакции срыва $^{12}C(d, n)^{13}N$ при лабораторном угле реакции $\theta_{lab} = 15°$ и энергии дейтронов $E_d = 12{,}9$ *Мэв*.

Для нейтронов, относящихся к энергетическому уровню 3,56 *Мэв* ядра ^{13}N была получена поляризация $-(0{,}39 \pm 0{,}11)$.